Math 210A, Fall 2017 (B. Li)

Hints / Sol'ns to HW #1

1 (1) Yes. Pf. $\forall \varepsilon > 0\ \exists N.\ n \geq N \Rightarrow |a_n - a| < \varepsilon$.
Assume M numbers are chosen and inserted into $\{a_n\}$. Label the new seq. as $\{b_n\}$.
~~Then~~. Assume those M numbers are now some b_n's with $n \leq M_0$ for some M_0.
This is true since there are only finitely many such numbers, then $n \geq N + M_0$
$\Rightarrow b_n = a_s$ with $s > N$. $\Rightarrow |b_n - a| = |a_s - a| < \varepsilon$.
Hence, $\lim_{n\to\infty} b_n = a$.

(2) No. $b_n = (-1)^{n-1}\frac{1}{n}$, $n = 1, 2, \cdots$.

(3) Yes. Let $\varepsilon = \frac{1}{4}$. Since $\lim_{n\to\infty} c_n = c$.
$\exists N$ s.t. $m, n \geq N$ $|c_m - c_n| < \varepsilon = \frac{1}{4}$ i.e., $c_m - c_n = 0$
since c_m and c_n are integers. Hence,
$c_n = c_N$ $\forall n \geq N$. ~~and~~ $\cancel{\Rightarrow |c_n - c| < \frac{1}{4}}$
So, $\lim_{n\to\infty} c_n = c_N$ is an integer. $c = c_N$.
(uniqueness).
Or: $\forall \varepsilon > 0$. $\exists M^{\geq N}$. $n \geq M \Rightarrow |c_n - c| < \varepsilon$.
But $c_n = c_N$. So, $|c_N - c| < \varepsilon$. Hence $c = c_N$.

(4) No. $a_n = -b_n = n$.

2. (1) $\lim_{x\to 0}\frac{\sin x}{x}=1 \Longrightarrow \lim_{n\to\infty} n\sin\frac{1}{n}=1$.

(2) $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{-n}=\lim_{n\to\infty}\left[\left(1+\frac{1}{2n}\right)^{2n}\right]^{\left(-\frac{1}{2}\right)}=e^{-\frac{1}{2}}=\frac{1}{\sqrt{e}}$

(3) $\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}\quad\left(\frac{0}{0}\right)$

$=\lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x}=\lim_{x\to 0}\frac{\frac{x}{1+x}}{2x}=\frac{1}{2}$

3. If $a_n\to a$, then $a_{n_k}\to a$ (as $k\to\infty$)

pf $\forall\varepsilon>0$. $a_n\to a\Longrightarrow \exists N.\ni n\ge N$
$\Longrightarrow |a_n-a|<\varepsilon$.
For this N, $k\ge N\Longrightarrow n_k\ge k\ge N$
$\Longrightarrow |a_{n_k}-a|<\varepsilon$. Hence $a_{n_k}\to a$.

If any subseq. of $\{a_n\}$ converges then $\{a_n\}$ is itself a subseq of $\{a_n\}$. So it converges.

Try this: Show that $\{a_n\}$ converges if and only any subseq of $\{a_n\}$ has a further subseq that converges.

HW#1. <u>Solution</u> (Math 210A, Fall 2017)

4. $$\sum_{n=p}^{q} a_n b_n \underset{a_n = A_n - A_{n-1}}{=} \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$$

$$= A_p b_p + A_{p+1} b_{p+1} + \cdots + A_{q-1} b_{q-1} + \underline{A_q b_q}$$
$$- \underline{\underline{A_{p-1} b_p}} - A_p b_{p+1} - \cdots - A_{q-1} b_q$$

$$= A_p (b_p - b_{p+1}) + A_{p+1} (b_{p+1} - b_{p+2})$$
$$+ \cdots + A_{q-1}(b_{q-1} - b_q) + \underline{A_q b_q} - \underline{\underline{A_{p-1} b_p}}$$

$$= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p .$$

5. (1) $$\frac{2}{(2k-1)(2k+1)} = \frac{1}{2k-1} - \frac{1}{2k+1}$$

$$\sum_{k=1}^{N} \cdots = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2N-1} - \frac{1}{2N+1}\right)$$

$$= 1 - \frac{1}{2N+1} \to 1.$$

So, $$\sum_{1}^{\infty} \frac{2}{(2k-1)(2k+1)} = 1$$

(2) ~~(b)~~ $$\sum_{k=0}^{\infty} \frac{k+1}{k!} = \sum_{k=0}^{\infty} \frac{k}{k!} + \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{k!}}_{=e}$$

$$\overset{j=k-1}{=} \sum_{j=0}^{\infty} \frac{1}{j!} + e$$

$$= e + e = 2e$$

6. (1) Divergence. Positive series.

$$\frac{k^2+k-1}{k^3+4k^2+10} \Big/ \frac{1}{k} \to 1. \quad \sum \frac{1}{k} \text{ diverges.}$$

(2) Converges but not absolutely convergent.

$$\frac{(-1)^k k^2+k-1}{k^3+k^2+10} = \frac{(-1)^k k^2}{k^3+k^2+10} + \underbrace{\frac{k-1}{k^3+k^2+10}}_{\text{positive}}$$

$(-1)^k c_k$. $\quad c_k = \dfrac{k^2}{k^3+k^2+10} > 0$

$$\frac{d}{dk} c_k = \frac{-k^2(3k^2+2k) \cancel{-k^2} + 2k(k^3+k^2+10)}{(\qquad)^2}$$

$$= \frac{-k^4+\cdots}{(\qquad)^2} < 0 \quad \text{for } k \gg 1.$$

$$\frac{k-1}{k^3+k^2+10} \Big/ \frac{1}{k^2} \to 1$$

$\sum \frac{1}{k^2}$ converges

So, $c_k \downarrow 0$. $(k \geq k_0) \Longrightarrow \sum (-1)^k c_k$ converges

But $|c_k| = O(\frac{1}{k})$. $\sum |c_k|$ diverges.

(3) $k \sin(\frac{1}{k}) = \dfrac{\sin \frac{1}{k}}{\frac{1}{k}} \to 1$ as $k \to \infty$

So, diverges.

(4) $\frac{1}{k} - \ln(1+\frac{1}{k}) = \frac{1}{k} - \left[\frac{1}{k} - \frac{1}{k^2} + O(\frac{1}{k^3})\right]$

$$= \frac{1}{k^2} + O(\frac{1}{k^3})$$

$0 < x < 1 \Longrightarrow x > \ln(1+x)$.

So, $\frac{1}{k} - \ln(1+\frac{1}{k}) > 0$.

and $\dfrac{\frac{1}{k} - \ln(1+\frac{1}{k})}{\frac{1}{k^2}} \to 1$

So, $\sum k^{-2}$ converges $\Longrightarrow \sum \left[\frac{1}{k} - \ln(1+\frac{1}{k})\right]$ converges.