Math 210A. Fall 2017 (B. Li)

Hints / Sol'ns to HW #2

1. (1) False. $a_n = \frac{1}{n}$, $b_n = -\frac{1}{n}$ $(n=1,2,\cdots)$.

(2) True. Assume $\sum_{n=1}^{\infty} a_n = A$ (convergence).
Then $$\sum_{k=1}^{n} b_k = \sum_{k=1}^{2n} a_k \longrightarrow A \quad \text{as } n\to\infty.$$
Hence $\sum_1^\infty b_n = A$ (convergence).

(3) False. $a_n = (-1)^n \frac{1}{\sqrt{n}}$. $b_n = \frac{1}{\sqrt{n}}$. $(n=1,2,\cdots)$

(4) True. Pf. Suppose $\sum_1^\infty a_{2n-1} = a_1 + a_3 + a_5 + \cdots$ converges. Then
$$a_1 + 0 + a_3 + 0 + a_5 + 0 + \cdots \quad \text{converges}$$
But $a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$ converges (by the assumption). Hence the difference
$$a_2 + a_4 + \cdots$$
Converges. Thus $\sum_1^\infty (-1)^{n-1} a_n$ converges absolutely, a contradiction. So $\sum a_{2n-1}$ diverges. Similarly, $\sum a_{2n}$ diverges.

2. See Lecture Notes. §1.2. p. 8.

3. No. Since for $a_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1, \quad \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1.$$

and for $b_n = \frac{1}{n^2}$ $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1$, $\lim_{n\to\infty} \sqrt[n]{|b_n|} = 1$.

But. Σa_n diverges. Σb_n converges.

4. (1) Yes. use the ratio test

(2) Yes. use the test for alternating series.

(3) ~~No~~ Yes. $\sin(k\pi + \frac{1}{k}) = (-1)^{k-1} \sin \frac{1}{k}$.

Use the test for alternating series.

5. $P_k = (1-\frac{1}{2^2})(1-\frac{1}{3^2}) \cdots (1-\frac{1}{k^2})(1-\frac{1}{(k+1)^2})$

$$= \frac{2^2-1}{2^2} \cdot \frac{3^2-1}{3^2} \cdot \frac{4^2-1}{4^2} \cdots \frac{k^2-1}{k^2} \cdot \frac{(k+1)^2-1}{(k+1)^2}$$

$$= \frac{1\cdot 3}{2^2} \cdot \frac{2\cdot 4}{3^2} \cdot \frac{3\cdot 5}{4^2} \cdots \frac{(k-1)(k+1)}{k^2} \cdot \frac{k(k+2)}{(k+1)^2}$$

$$= \frac{1}{2} \frac{k+2}{k+1} \longrightarrow \frac{1}{2}.$$

$$\tan\varphi_k = -\frac{1}{k^2} r\sin\theta \Big/ \left(1-\frac{1}{k^2} r\cos\theta\right)$$

$$= -\frac{r\sin\theta}{k^2 - r\cos\theta} \to 0 \text{ as } k \to \infty.$$

So $\varphi_k \to 0$.

Moreover, $\tan\varphi_k = O(\frac{1}{k^2})$ So, $\varphi_k = O(\frac{1}{k^2})$

Thus, $\sum_1^\infty \varphi_k$ converges. i.e. $\prod_{k=1}^\infty e^{i\varphi_k}$ converges.

($\sum_1^\infty \varphi_k \neq 0$ as this is positive (or negative) series, after finite terms.)

<u>Conclusion</u> $\prod_{k=1}^\infty \left(1-\frac{z^2}{k^2}\right)$ converges.

HW#2

6. (1) $\sum_0^\infty P_n = 1 \Rightarrow A\sum_0^\infty e^{-E_n/k_BT} = 1$

$E_n = (n+1/2)h\nu$

So, $\sum_0^\infty e^{-(n+\frac{1}{2})h\nu/k_BT} = A^{-1}$

$$A^{-1} = \sum_{n=0}^\infty e^{-n\alpha}\cdot e^{-\frac{\alpha}{2}} = e^{-\frac{\alpha}{2}}\sum_{n=0}^\infty e^{-\alpha n}$$

$\alpha = h\nu/k_BT > 0$.

$$e^{\alpha/2}A^{-1} = \sum_{n=0}^\infty e^{-\alpha n} = \frac{1}{1-e^{-\alpha}} = \frac{e^\alpha}{e^\alpha - 1}$$

$$A^{-1} = \frac{e^{\alpha/2}}{e^\alpha-1} \qquad A = \frac{e^\alpha - 1}{e^{\alpha/2}} \quad (\alpha = h\nu/k_BT)$$

(2) $\langle E(T)\rangle = \sum_{n=0}^\infty P_n E_n = \sum_{n=0}^\infty A e^{-E_n/k_BT}(n+\frac{1}{2})h\nu$

$$= Ah\nu \sum_{n=0}^{\infty} n e^{-(n+1/2)h\nu/k_BT} + \underbrace{\tfrac{1}{2}h\nu}_{hg(a)}$$

$$= Ah\nu e^{-h\nu/2k_BT} \sum_{n=0}^{\infty} n e^{-n\alpha} + \tfrac{1}{2}h\nu$$

$$= \frac{e^{\alpha}-1}{e^{\alpha/2}} h\nu e^{-1/2\cdot\alpha} \sum_{n=1}^{\infty} n(e^{-\alpha})^n + \tfrac{1}{2}h\nu$$

Let $q = e^{-\alpha} < 1$. ($q > 0$).

$$S_N = q + 2q^2 + \cdots + Nq^N$$

$$\frac{1}{q} S_N = 1 + 2q + \cdots + Nq^{N-1}$$

$$\frac{1}{q} S_N - S_N = 1 + q + q^2 + q^{N-1} - \underbrace{Nq^N}_{\to 0 \text{ as } N \to \infty}$$

$$\lim_{N\to\infty} \frac{1}{q} S_N = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$

$$\sum_{1}^{\infty} n(e^{-\alpha})^n = \frac{q}{1-q} = \frac{e^{-\alpha}}{1-e^{-\alpha}} = \frac{1}{e^{\alpha}-1}$$

$$\langle E(T)\rangle = \frac{e^{\alpha}-1}{e^{\alpha/2}} h\nu e^{-\alpha/2} \frac{1}{e^{\alpha}-1} + \tfrac{1}{2}h\nu$$

$$= h\nu e^{-\alpha} + \tfrac{1}{2}h\nu$$

$$= h\nu e^{-h\nu/k_BT} + \tfrac{1}{2}h\nu$$

I hope this is correct.