1. (a) \[ \lim_{n \to \infty} f_n(x) = f(x) = \frac{1}{x} \]

Let \( 0 < x \leq 1 \). If \( \frac{1}{n} < x \) then \( f_n(x) = 0 \). Hence \( \lim_{n \to \infty} f_n(x) = 0 \).

(b) If \( \{f_n\} \) converges in \( L^1(0,1) \), then \( \{\|f_n\| \} \) converges, where \( \|f_n\| = \sqrt{\int_0^1 |f_n(x)|^2 \, dx} \)

But \( \|f_n\|^2 = \int_0^1 |f_n(x)|^2 \, dx = \int_0^1 \left( \frac{1}{n} \right)^2 \, dx \)

\[ = - \frac{1}{x} \frac{1}{n} = -1 + n \to +\infty \text{ as } n \to +\infty. \]

So \( \{\|f_n\|\} \) does not converge. Hence \( \{f_n\} \) does not converge in \( L^1(0,1) \).

3. \[-\int_0^1 H(x) \phi(x) \, dx = -\int_0^A H(x) \phi(x) \, dx \]

\[ = -\int_0^A H(x) \phi(x) \, dx - \int_A^1 H(x) \phi(x) \, dx \]

\[ = -\int_A^1 \phi(x) \, dx = -\left( \phi(A) - \phi(1) \right) = \phi(1). \]

This equation (for any \( \phi \)) defines \( H'(x) = \delta(x) \).

2. (1) \( \int_0^1 |f(x)| \, dx \leq \sqrt{\int_0^1 |f(x)|^2 \, dx} \leq \sqrt{\int_0^1 1 \, dx} = 1. \)

(2) \( \|f_n - f\|_{L^1(0,1)} = \int_0^1 |f_n - f| \, dx \leq \sqrt{\int_0^1 |f_n - f|^2 \, dx} = \|f_n - f\|_{L^2(0,1)}. \)

4. (1) No. See function in Prob. #1.

(2) Yes. \( |h_n(x)| \leq M \) for all \( n \) and \( x \in I \).
5. By the Weierstrass Thm, there exist a sequence of polynomials \( P_0, P_1, P_2, \ldots \) such that

\[
\lim_{n \to \infty} \max_{x \in [0,1]} |P_n(x) - u(x)| = 0.
\]

Since \( \int_0^1 x^n u(x) \, dx = 0 \) for all \( n = 0, 1, 2, \ldots \)

\[
\int_0^1 x^n u(x) \, dx = 0 \quad \text{for all } n = 1, 2, \ldots
\]

Therefore,

\[
\int_0^1 u(x) \, dx = \int_0^1 \left[ u(x) - P_n(x) \right] \, dx + \int_0^1 u(x) P_n(x) \, dx
\]

\[
= \int_0^1 \left| u(x) \right| \max_{x \in [0,1]} |u(x) - P_n(x)| \, dx
\]

\[
= a_n \int_0^1 |u(x)| \, dx \to 0 \quad \text{as } n \to \infty
\]

Hence, \( \int_0^1 u(x) \, dx = 0 \). \( u \) is continuous.

So, \( u(0) = 0 \) (by \( f \) is continuous).

HW #4

Part 5. PF \( \int_0^1 [f' g - f g'] \, dx + \int_0^1 [f + g] \, dx = \frac{\partial}{\partial x} \int_0^1 [f g] \, dx \)

\[
\int_0^1 |f(x) - f_0| \, dx
\]

\[
= -\int_0^1 (f(x) + f_0) \, dx + 2\int_0^1 |f(x)| \, dx + 2\int_0^1 |f_0| \, dx
\]

\[
= -\int_0^1 f(x) \, dx - 2\int_0^1 f_0 \, dx - \int_0^1 |f_0| \, dx + 2\int_0^1 |f_0| \, dx
\]

\[
= -2\int_0^1 f_0 \, dx + \int_0^1 |f(x)| \, dx + \int_0^1 |f_0| \, dx
\]

\[
\to -2\int_0^1 f_0^2 \, dx + \int_0^1 |f'| \, dx + \int_0^1 |f_0'| \, dx = 0
\]

So, \( \int_0^1 |f(x) - f_0|^2 \, dx \to 0 \) as \( n \to \infty \). Q.E.D.