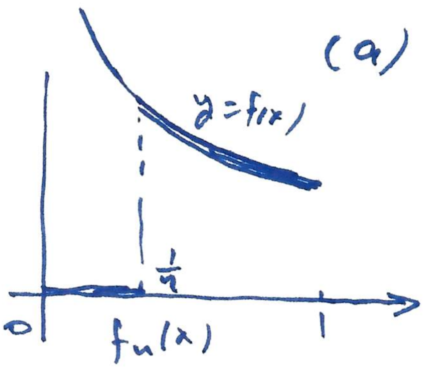
Math 210A Fall 2017 (B. Li)

HW#3 Hints/Solutions (Math 210A, Fall 2017)

1.



(a) $\lim_{n\to\infty} f_n(0) = 0 = f(0)$

Let $0 < x \le 1$. If $\frac{1}{n} < x$ then $f_n(x) = f(x)$.

Hence $\lim_{n\to\infty} f_n(x) = f(x)$.

(b) If $\{f_n\}$ converges in $L^2(0,1)$, then $\{\|f_n\|\}$ converges, where $\|f_n\| = \sqrt{\int_0^1 |f_n(x)|^2 dx}$

But $\|f_n\|^2 = \int_0^1 |f_n(x)|^2 dx = \int_{\frac{1}{n}}^1 \left(\frac{1}{x}\right)^2 dx$

$= -\frac{1}{x}\Big|_{\frac{1}{n}}^1 = -1 + n \longrightarrow +\infty$ as $n \to \infty$.

So $\{\|f_n\|\}$ does not converge. Hence $\{f_n\}$ does not converge in $L^2(0,1)$.

3. $-\int_{-\infty}^{\infty} H(x)\phi'(x)dx = -\int_{-A}^{A} H(x)\phi'(x)dx$

$= -\int_{-A}^{0} H(x)\phi'(x)dx - \int_0^A H(x)\phi'(x)dx$

$= -\int_0^A \phi'(x)dx = -[\underbrace{\phi(A)}_{\phi(A)=0} - \phi(0)] = \phi(0)$.

This equation (for any ϕ) defines $H'(x) = \delta(x)$, — The Dirac δ-function.

2. (1) $\int_0^1 |f(x)|dx = \int_0^1 1\cdot|f(x)|dx \overset{\text{Cauchy-Schwarz}}{\le} \underbrace{\sqrt{\int_0^1 1^2 dx}}_{=1} \cdot \sqrt{\int_0^1 |f(x)|^2 dx}$

(2) $\|f_n - f\|_{L^1(0,1)} = \int_0^1 |f_n - f|dx \le \sqrt{\int_0^1 |f_n - f|^2 dx} = \|f_n - f\|_{L^2(0,1)} \to 0$.

4. (1) No. See the function in Prob. #1.

(2) Yes. $|h_n(x)| \le M$ $(n = 1, 2, \dots, x \in I)$. $\lim_{n\to\infty} |h_n(x)| = |h(x)| \le M$ $\forall x \in I$.

HW#3.

5. By the Weierstrass Thm. there exist a sequence of polynomials $p_1, p_2, p_3, \dots$ such that

$$a_n \equiv \max_{x\in[0,1]} |p_n(x) - u(x)| \to 0 \text{ as } n\to\infty.$$

Since $\int_0^1 x^n u(x)\,dx = 0$ for all $n = 0, 1, 2, \dots$

$$\int_0^1 p_k(x)u(x)\,dx = 0 \text{ for all } k = 1, 2, \dots.$$

Thus.
$$\int_0^1 u(x)^2 dx = \int_0^1 u(x)\cdot[u(x) - p_k(x)]\,dx + \underbrace{\int_0^1 u(x)p_k(x)\,dx}_{=0}$$

$$\le \int_0^1 |u(x)| \underbrace{\max_{x\in[0,1]} |u(x) - p_k(x)|}_{\equiv a_k} dx$$

$$= a_k \int_0^1 |u(x)|\,dx \longrightarrow 0 \text{ as } k\to\infty$$

Hence $\int_0^1 u(x)^2 dx = 0$. u is continuous.
So, $u(x) = 0$ $\forall x\in[0,1]$.

Q.E.D.

HW#4
Part of 5. Pf
$$\int_0^1 |f_k - f|^2 dx + \int_0^1 |f_k + f|^2 dx = 2\int_0^1 |f_k|^2 dx + 2\int_0^1 |f|^2 dx$$

Law of Parallelogram

So
$$0 \le \int_0^1 |f_k - f|^2 dx$$
$$= -\int_0^1 |f_k + f|^2 dx + 2\int_0^1 |f_k|^2 dx + 2\int_0^1 |f|^2 dx$$
$$= -\int_0^1 |f_k|^2 dx - 2\int_0^1 f_k f\,dx - \int_0^1 |f|^2 dx + 2\int_0^1 |f_k|^2 dx + 2\int_0^1 |f|^2 dx$$
$$= -2\int_0^1 f_k f\,dx + \int_0^1 |f_k|^2 dx + \int_0^1 |f|^2 dx$$
$$\longrightarrow -2\int_0^1 f^2 dx + \int_0^1 |f|^2 dx + \int_0^1 |f|^2 dx = 0$$

So, $\int_0^1 |f_k - f|^2 dx \to 0$ as $k\to\infty$. Q.E.D.