Math 210A, Fall 2017 (B. Li)

Hints/Soln's to HW #4

1. (1) No. $\int_{-\infty}^{\infty} 1^2\,dx = +\infty$.

(2) Yes. $\forall g \in L^2(I)$, then

$$\left|\int_I f_n g\,dx - \int_I f g\,dx\right| \le \int_I |f_n - f|\,|g|\,dx$$

$$\overset{\text{Cauchy-Schwarz}}{\le} \sqrt{\int_I |f_n - f|^2 dx}\,\sqrt{\int_I |g|^2 dx}$$

$$\overset{\text{notation}}{=} \|f_n - f\|_{L^2(I)}\,\|g\|_{L^2(I)} \longrightarrow 0 \quad \text{since } f_n \to f \text{ in } L^2.$$

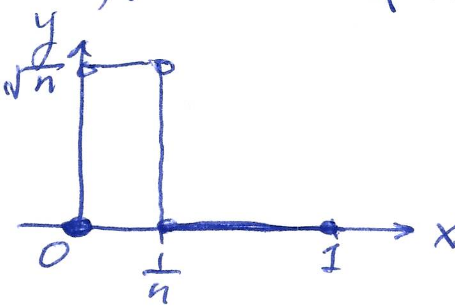
(3) False. $f(x) = 0$. $f_n(0) = 1$. $f_n(x) = \sqrt{n}$ $x \in (0, \frac{1}{n})$

$f_n(x) = 0$ $x \in [\frac{1}{n}, 1]$.

Then $f_n(x) \to f(x) = 0$ $\forall x \in [0,1]$.

But $f_n \not\to f$ in $L^2(0,1)$

since $\|f_n\| = 1$ $\forall n = 1, 2, \cdots$.



(4) False.

2. Pf. Suppose $f_n \to f$ in $L^2(0,1)$. Then $f_n \to f$ weakly in $L^2(0,1)$, cf. Prob. #1 (2).

Also, $\|f_n - f\|_{L^2(0,1)} \to 0 \Longrightarrow \|f_n - f\|^2$

$$= \|f_n\|^2 + \|f\|^2 - \int_0^1 2 f_n f\,dx \longrightarrow 0$$

But $\int_0^1 f_n f\,dx \to \int_0^1 |f|^2 dx = \|f\|^2$ (weak convergence)

Hence

$$\|f_n\|^2 = \|f_n - f\|^2 - \|f\|^2 + \int_0^1 2 f_n f\,dx$$
$$\longrightarrow 0 - \|f\|^2 + 2\|f\|^2 = \|f\|^2.$$

A more general pf.

$$\big|\, \|f_n\| - \|f\| \,\big| \le \|f_n - f\| \longrightarrow 0.$$

Why ↑ ?

Conversely, assume $\|f_n\| \to \|f\|$, $f_n \to f$ weakly.
then

$$\|f_n - f\|^2 = \|f_n\|^2 + \|f\|^2 - 2\int_0^1 f_n f\,dx$$
$$\longrightarrow \|f\|^2 + \|f\|^2 - 2\int_0^1 |f|^2 dx = 0.$$

3. Pf $I_n \equiv \left| \int_{-1}^1 f_n(x)\phi(x) - \phi(0) \right| = \left| \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(x) \, dx - \phi(0) \right|$

$= \left| \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} [\phi(x) - \phi(0)]\,dx \right| \le \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} |\phi(x) - \phi(0)|\,dx$

$\forall \varepsilon > 0$. ϕ is cont. at 0. $\Longrightarrow \exists\, \delta > 0$. s.t. $|x| < \delta$
$\Longrightarrow |\phi(x) - \phi(0)| < \varepsilon$. Let $N > \frac{1}{\delta}$. then
$n \ge N \Longrightarrow \frac{1}{n} < \delta$. So, $-\frac{1}{n} \le x \le \frac{1}{n} \Longrightarrow |x| < \delta$
$\Longrightarrow |\phi(x) - \phi(0)| < \varepsilon$. Hence, $n \ge N \Longrightarrow I_n < \varepsilon$.

4. Let $f_n(x) = S_n(x) - S_{n-1}(x)$ $(n = 1, 2, \cdots \;\; S_0(x) \equiv 0)$

$S_n(x) = n^2 x(1-x^2)^n$ see Lecture notes.
Section 1.3 ~~Section~~ P. 2.

5. ~~(a)~~ (1) $a_k = \left| \frac{(-1)^k k}{3^{k+1}} (z-4)^{2k} \right| = \frac{k}{3^{k+1}} |z-4|^{2k}$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{k+1}{3^{k+2}} |z-4|^{2k+2} \Big/ \frac{k}{3^{k+1}} |z-4|^{2k}$$

$$= \lim_{k\to\infty} \left(\frac{1}{3} |z-4|^2 \cdot \frac{k+1}{k} \right)$$

$$= \frac{1}{3} |z-4|^2 < 1$$

$|z-4| < \sqrt{3}$. So, the radius of convergence is $R = \sqrt{3}$.

~~(b)~~ (2) For $\sum_{k=1}^{\infty} \frac{z^k}{k}$: $a_k = \frac{|z|^k}{k}$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{|z|^{k+1}}{k+1} \Big/ \frac{|z|^k}{k} = |z| < 1.$$

So, $R_1 = 1$

For $\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} z^k$: $b_k = \left| \frac{(-1)^k}{2^k} z^k \right| = \frac{|z|^k}{2^k}$

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \lim_{k\to\infty} \left(\frac{|z|^{k+1}}{2^{k+1}} \Big/ \frac{|z|^k}{2^k} \right) = \frac{|z|}{2} < 1$$

$|z| < 2$. $R_2 = 2$.

So, the radius of convergence is $R = 1$.

#6. (Not exactly the same).

4. ~~(a)~~ (1) $\sum_{k=1}^{\infty} \frac{1}{k^2 3^k} x^k = \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\frac{x}{3}\right)^k = \sum_{k=1}^{\infty} \frac{1}{k^2} z^k = g(z)$

$z = \frac{x}{3}$. $|z|<1$. $g'(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{k-1} = \frac{1}{z} \sum_{k=1}^{\infty} \frac{1}{k} z^k = \frac{1}{z} f(z)$

$f(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^k$ $f(0) = 0$

$f'(z) = \sum_{k=1}^{\infty} z^{k-1} = \frac{1}{1-z}$. $f(z) = \int_0^z \frac{1}{1-z} dz + c$

$f(0) = 0 \Rightarrow c = 0 \Rightarrow f(z) = \int_0^z \frac{1}{1-z} dz = \ln(1-z) \Big|_0^z$

$= \ln(1-z)$

$$g'(z) = \frac{1}{z}\ln(1-z).$$

$$g(z) = \int_0^z \frac{1}{y}\ln(1-y)\,dy + C_1$$

$$g(0) = 0 \Rightarrow C_1 = 0.$$

$$g(z) = \int_0^z \frac{\ln(1-y)}{y}\,dy$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2 3^k} x^k = \int_0^{x/3} \frac{\ln(1-y)}{y}\,dy.$$

(2) $\sum_{k=1}^{\infty} \frac{k^2}{3^k} x^k = \sum_{k=1}^{\infty} k^2 \left(\frac{x}{3}\right)^k = \frac{x}{3}\sum_{k=1}^{\infty} k^2 \left(\frac{x}{3}\right)^{k-1}$

$$f(z) = \sum_{k=1}^{\infty} k^2 z^{k-1}$$

$$\int_0^z f(z)\,dz = \sum_{k=1}^{\infty} k z^k$$

$$S(z) = \sum_{k=1}^{\infty} k z^k = z + 2z^2 + 3z^3 + \cdots$$

$$\frac{1}{z} S(z) = 1 + 2z + 3z^2 + \cdots$$

$$\frac{1}{z} S(z) - S(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}$$

$$S(z)\left(\frac{1}{z} - 1\right) = \frac{1}{1-z}, \quad S(z) = \frac{z}{(1-z)^2}$$

$$\int_0^z f(z)\,dz = \frac{z}{(1-z)^2}$$

$$f(z) = \left(\frac{z}{(1-z)^2}\right)' = -\frac{z+1}{(z-1)^3}$$

$$\sum_{k=1}^{\infty} \frac{k^2}{3^k} x^k = \frac{x}{3} f\left(\frac{x}{3}\right) = \frac{x}{3}(-1)\frac{\frac{x}{3}+1}{\left(\frac{x}{3}-1\right)^3}$$

$$= -\frac{x(x+3)}{9(x-3)^3 \cdot \left(\frac{1}{3}\right)^3} = -\frac{3x(x+3)}{(x-3)^3}.$$