Math 210A, Fall 2017 (B. Li)

Hints / Sol'ns to HW # 6

1. (1) True $A\vec{u} = \lambda\vec{u}$, $\vec{u} \neq \vec{0}$.
$\det(\lambda I - A) = 0$. So, if $\lambda = 0$, then $\det(-A) = 0$
So, $\det A = 0$
[If A is $n\times n$, then $\det(-A) = (-1)^n \det A$.]

(2) False. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. $\lambda_1 = 1$. $\lambda_2 = 2$.

(3) True. $A\vec{u} = \lambda\vec{u}$ $(\vec{u} \neq \vec{0})$. $-7I\vec{u} = -7\vec{u}$.
So, $A\vec{u} - 7I\vec{u} = \lambda\vec{u} - 7\vec{u}$
$(A - 7I)\vec{u} = (\lambda - 7)\vec{u}$, $\vec{u} \neq \vec{0}$.

(4) True. $\forall x \neq 0$. $x^T C^T C x = (x^T C^T) C x$
$= (Cx)^T Cx = \|Cx\|^2 \geq 0$ If "$=0$"
then $Cx = 0$, $x = 0$. $C^T C$ is obviously symmetric. $(C^T C)^T = C^T (C^T)^T = C^T C$.

2. Solution A can be diagonalized.
$A = P^{-1} \Lambda P$ $\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.
(1) Rank $A = 2$.
(2) $\det A = 0 = \lambda_1 \lambda_2 \lambda_3$ $\lambda_1 = 0$. $\lambda_2 = 1$. $\lambda_3 = 2$
(3) $A^T A$ is uncertain. It may have eigenvalues $\lambda_1^2 = 0$, $\lambda_2^2 = 1$, $\lambda_3^2 = 4$ (in particular, if A is symmetric).

It may also have different eigenvalues
For example, if $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Then

$A^TA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$: eigenvales: $0, 2, 4$

not: $0, 1, 4$.

(4) $\frac{1}{\lambda+1}$: $1, \frac{1}{2}, \frac{1}{3}$.

Pf $A\vec{u} = \lambda \vec{u}$ $\vec{u} \neq \vec{0}$. $(A+I)\vec{u} = (\lambda+1)\vec{u}$.

$A+I$ has eigenvales $\lambda+1 = 1, 2, 3$ all >0.

So, $\det(A+I) \neq 0$.

$$\vec{u} = (\lambda+1)(A+I)^{-1}\vec{u}$$

$$(A+I)^{-1}\vec{u} = \frac{1}{\lambda+1}\vec{u} \quad (\vec{u} \neq \vec{0}).$$

3. $\det(\lambda I - A) = \lambda^2 - (a_{11}+a_{22})\lambda - a_{12}a_{21} + a_{11}a_{22}$

$= \lambda^2 - (\mathrm{tr}A)\lambda + \det A$ $\quad A = (a_{ij})$

$= (\lambda - \lambda_1)(\lambda - \lambda_2)$

$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$.

So, $\lambda_1 + \lambda_2 = \mathrm{tr}A = 1$, $\lambda_1\lambda_2 = \det A = 2$

$\lambda_1^2 + \lambda_2^2 = (\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2 = 1 - 4 = -3$.

(λ_1, λ_2: complex values).

4. $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ $\quad \lambda I - A = \begin{bmatrix} \lambda-4 & -3 \\ -1 & \lambda-2 \end{bmatrix}$

$\det(\lambda I - A) = 0$ $\quad (\lambda-4)(\lambda-2) - 3 = 0$

$\lambda^2 - 6\lambda + 5 = 0$ $\quad \lambda_1 = 1$ $\quad \lambda_2 = 5$

$\lambda_1 = 1$: $\lambda_1 I - A = \begin{bmatrix} -3 & -3 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 + x_2 = 0$, $x_2 = -x_1$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $\quad \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 5 \quad \lambda_2 I - A = \begin{bmatrix} 1 & -3 \\ -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$S = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = [\vec{u}_1\ \vec{u}_2] \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

$S^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$

$A = S \Lambda S^{-1}$

$A^{100} = S \Lambda^{100} S^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$

$= \frac{1}{4} \begin{bmatrix} 1 + 3\cdot 5^{100} & -3 + 3\cdot 5^{100} \\ -1 + 5^{100} & 3 + 5^{100} \end{bmatrix}$

5. Let $e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$ Then $Ae = \begin{bmatrix} p_{ij} \end{bmatrix} e$

$(Ae)_i = \sum_{j=1}^{n} p_{ij} e_j = \sum_{j=1}^{n} p_{ij} = 1, \quad i = 1, 2, \cdots, n$

$e \neq 0$. So, e is an eigenvector corresponding to the eigen value 1.

6. Let A be this matrix. Clearly A is real and symmetric.

$2 > 0 \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$

$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 3 & -2 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = 6 - 2 = 4 > 0$

$$A = \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 3 & -2 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 4 & -3 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -3 \\ -1 & 2 \end{vmatrix} = 8-3 = 5 > 0$$

So A is SPD.

Note: there is a pattern

$$|2| = 2 \quad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 \quad \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 4$$

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 5.$$

7. Clearly $a_{ij} = a_{ji}$. So, A is (real) symmetric

$\forall a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \in R^{n+1}$.

$$a \cdot Aa = \sum_{i,j=0}^{n} a_{ij} a_i a_j = \sum_{i,j=0}^{n} a_i a_j \int_0^1 x^{i+j} f(x) dx$$

$$= \sum_{i=0}^{n} a_i \int_0^1 x^i \Big(\sum_{j=0}^{n} a_j x^j \Big) f(x) dx$$

$$= \int_0^1 \Big(\sum_{i=0}^{n} a_i x^i \Big) \Big(\sum_{j=0}^{n} a_j x^j \Big) f(x) dx$$

$$= \int_0^1 \Big| \sum_{i=0}^{n} a_i x^i \Big|^2 f(x) dx \geq 0$$

If this is 0, then $\sum_{i=0}^{n} a_i x^i = 0$ on $[0,1]$

So, all $a_i = 0$ $(i = 0, 1, 2, \cdots n)$.

Hence A is SPD.