Math 210 A. Fall 2017 (B. Li)

Hints / Sol'ns to HW # 7

1. (1) $f(x,y) = \frac{1}{4}x^4 + x^2 y + y^2 \qquad \partial_x f = x^3 + 2xy$

$\partial_{xx}^2 f = 3x^2 + 2y \qquad \partial_y f = x^2 + 2y \qquad \partial_{yy} f = 2$

$\partial_{xy} f = \partial_{yx} f = 2x$

$$\text{Hessian} = \begin{bmatrix} \partial_{xx} f & \partial_{xy} f \\ \partial_{yx} f & \partial_{yy} f \end{bmatrix} = \begin{bmatrix} 3x^2 + 2y & 2x \\ 2x & 2 \end{bmatrix}$$

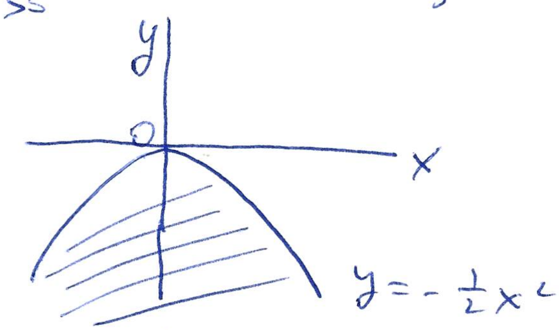
$\det = 2(3x^2 + 2y) - 4x^2 = 2x^2 + 4y.$

positive def. $\iff \begin{cases} 3x^2 + 2y > 0 \\ 2x^2 + 4y > 0 \end{cases} \iff \begin{cases} x^2 + \frac{2}{3}y > 0 \\ x^2 + 2y > 0 \end{cases}$

$\cancel{\text{Or}} \iff \begin{cases} x^2 + \frac{2}{3}y > 0 \\ \frac{1}{3}x^2 + \frac{2}{3}y > 0 \end{cases} \iff \frac{1}{3}x^2 + \frac{2}{3}y > 0$

$\iff x^2 + 2y > 0$

Hessian is SPD $\iff \boxed{y > -\frac{1}{2}x^2}$



(2) Let $y_1 = \frac{1}{\sqrt{6}}(x_1 - x_2 + 2x_3)$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = Q \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

choose 2 rows of Q (row 2 and row 3) so that Q is orthogonal: $Q^T Q = I$.

Then. $\vec{x} = Q^T \vec{y}$. $\vec{x}^T A \vec{x} = (Q^T \vec{y})^T A Q^T \vec{y}$

$= \vec{y}^T (Q A Q^T) \vec{y}$.

The new quadratic form is

$$\vec{x}^T A \vec{x} = 4(\sqrt{6}\, y_1)^2 = 24 y_1^2 \qquad QAQ^T = \begin{bmatrix} 24 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank A = rank $QAQ^T = 1$.

eigenvalues of A: $24, 0, 0$.

2. $x^T(A+B)x = x^T A x + x^T B x > 0$ if $x \neq 0$.

$\Longrightarrow A+B$ is SPD $\quad \boxed{(A+B)^T = A^T + B^T = A+B}$

$(\alpha A)^T = \alpha A^T = \alpha A$.

$x^T(\alpha A)x = \alpha x^T A x > 0 \quad \forall x \neq 0$.

$\Longrightarrow \alpha A$ is SPD

$(A^k)^T = {A^T}^k = A^k$. $\quad A^k$ is symmetric.

~~If~~ A ~~x~~ is SPD $\Longrightarrow A = C^T C \quad C$: nonsingular

$$x^T A^k x = x^T \underbrace{C^T C \cdot C^T C \cdots C^T C}_{k} x$$

If $k = 2n$ is even. $x^T A^k x = x^T A^{2n} x$

$= x^T (A^n)^T A^n x = (A^n x)^T A^n x = \|A^n x\| \geq 0$

$= 0 \Longleftrightarrow A^n x = 0 \Longleftrightarrow x = 0$.

If $k = 2n+1$ is odd $\quad A = C^T C$

$$x^T A^k x = x^T A^n A A^n x = (x^T (A^n)^T) C^T C (A^n x)$$

$= (A^n x)^T C^T C A^n x = (CA^n x)^T (CA^n x)$

$= \| CA^n x \| \geq 0 \quad$ "$= 0$" $\Longleftrightarrow CA^n x = 0 \Longleftrightarrow x = 0$.

So, A^k is SPD.

Since $(A^{-1})^T = (A^T)^{-1} = A^{-1}$, A^{-1} is symmetric.

A: SPD $\Rightarrow$ $A = C^T C$ C: nonsingular.

So, $A^{-1} = (C^T C)^{-1} = C^{-1}(C^T)^{-1} = C^{-1}(C^{-1})^T$

$= ((C^{-1})^T)^T ((C^{-1})^T) = B^T B$

with $B = (C^{-1})^T = C^{-T}$ nonsingular.

So, A^{-1} is SPD.

AB may not be symmetric: $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ A, B: SPD.

AB ~~=~~ $= \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}$. not symmetric.

3. (1) $A^T = A \Rightarrow B^T = (C^T A^T C)^T = C^T A^T (C^T)^T$

$= C^T A C = B$. So, B is symmetric.

$C^T A C = B \Rightarrow A = C^{-T} B C^{-1}$

Similarly, B is symmetric $\Rightarrow$ A is.

(2) A: SPD $\Rightarrow$ $x^T B x = x^T C^T A C x$

$= (Cx)^T A (Cx) > 0$ $\forall Cx \neq 0$

$\Leftrightarrow \forall x \neq 0$. $\Rightarrow$ B is SPD

Similary. B is SPD $\Rightarrow$ A is.

4. $(UV)^T UV = V^T U^T U V = V^T V = I$.

$\Rightarrow$ UV is orthogonal.

5. (1) $A = \begin{bmatrix} 5 & 1 \\ -2 & 2 \end{bmatrix}$ $\det(\lambda I - A) = 0$ $\lambda^2 - 7\lambda + 12 = 0$

$\lambda_1 = 3.$ $\lambda_2 = 4$

$\lambda_1 I - A = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ $\vec{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_2 I - A = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$S = [\vec{u}_1 \ \vec{u}_2] = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

$S^{-1} = \frac{1}{-1+2} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

$A = S\Lambda S^{-1} \Longrightarrow e^A = S e^{\Lambda} S^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} e^3 & 0 \\ 0 & e^4 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

$= \cdots$

(2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ $\cdots$

$e^A = I + A + \frac{A^2}{2!} + \cdots$

$= \begin{bmatrix} 1+1+\frac{1}{2!}+\cdots & 0+1+\frac{2}{2!}+\frac{3}{3!}+\cdots \\ 0 & 1+1+\frac{1}{2!}+\cdots \end{bmatrix}$

$= \begin{bmatrix} e & e \\ 0 & e \end{bmatrix} = eA.$

(3) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$e^A = I + A + \frac{A^2}{2!} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$

6. (1) $e^A = I + A + \frac{A^2}{2!} + \cdots = I + A$ all $A^k = 0$ $(k \geq 0)$

(2) $A^2 = A \Rightarrow A^3 = A^2 A = A^2 = A \Rightarrow$ all $A^k = A$ $k \geq 2$

$e^A = I + A + \frac{A^2}{2!} + \cdots = I + A + \frac{A}{2!} + \frac{A}{3!} + \cdots = I + (e-1)A.$

$(e = 1 + 1 + \frac{1}{2!} + \cdots)$

(13) We show in general $AB = BA \Rightarrow e^{A+B} = e^A e^B$

$$e^A e^B = \left(\sum_0^\infty \frac{1}{k!} A^k\right)\left(\sum_0^\infty \frac{1}{k!} B^k\right) = \sum_0^\infty C_k$$

$$C_k = \frac{1}{k!} A^k + \frac{1}{(k-1)!} A^{k-1} B + \frac{1}{(k-2)!} A^{k-2} B^2$$
$$+ \cdots + \frac{1}{(k-1)!} A B^{k-1} + \frac{1}{k!} B^k$$
$$= \frac{1}{k!}\left(A^k + \binom{k}{k-1} A^{k-1} B + \binom{k}{k-2} A^{k-2} B^2\right.$$
$$\left. + \cdots + B^k\right) = \frac{1}{k!}(A+B)^k.$$

7. $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = I + A + \frac{A}{2!} + \frac{A}{3!} + \cdots$$
$$= I + \cancel{A}\left(1 + \frac{1}{2!} + \cdots\right)A = I + \cancel{A}(e-1)A$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e-1)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e-1 & e-1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$e^B = I + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$
$$e^A e^B = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & -1 \\ 0 & 1 \end{bmatrix}$$
$$e^B e^A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e-2 \\ 0 & e \end{bmatrix}$$
$$e^B e^A \neq e^A e^B$$
$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$$

$e^A e^B$, $e^B e^A$, e^{A+B} : all different.

8. (1) $A = RU$ R: rotation, U: SPD

$$A^TA = U^2. \quad U = \sqrt{A^TA}$$

$$A^TA = \frac{1}{10}\begin{bmatrix} 10 & 0 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 100 & 60 \\ 60 & 100 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

$$\begin{vmatrix} \lambda-10 & -6 \\ -6 & \lambda-10 \end{vmatrix} = \lambda^2 - 20\lambda + 64 = 0 \quad (\lambda-4)(\lambda-16) = 0$$

$\lambda_1 = 4, \lambda_2 = 16$

$$\lambda_1 I - A^TA = \begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \vec{u}_1 \; \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 I - A^TA = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \vec{u}_2 \; \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^TA = \lambda_1 \vec{u}_1 \otimes \vec{u}_1 + \lambda_2 \vec{u}_2 \otimes \vec{u}_2$$

$$U = \sqrt{A^TA} = \sqrt{\lambda_1}\,\vec{u}_1 \otimes \vec{u}_1 + \sqrt{\lambda_2}\,\vec{u}_2 \otimes \vec{u}_2$$

$$= 2\,\vec{u}_1 \otimes \vec{u}_1 + 4\,\vec{u}_2 \otimes \vec{u}_2$$

$$= \frac{2}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{4}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$R = AU^{-1} = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{-1}$$

$$\cancel{= \frac{1}{\sqrt{10}}\begin{bmatrix} 36 & 13 \\ 8 & 24 \end{bmatrix}}$$

$$= \frac{1}{\sqrt{10}} \cdot \frac{1}{8}\begin{bmatrix} 10 & 6 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{8\sqrt{10}}\begin{bmatrix} 24 & -8 \\ -8 & 24 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

(2) $$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}$$

$A^TA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\lambda = 3, 1$. $\underline{\sigma_1 = \sqrt{3}.\ \sigma_2 = 1.}$

Col. of V: orthonormal eigenvectors of A^TA. $A^TA = V^T\Sigma^2 V$ V

$AA^T = U\Sigma^2 U^T$. In fact $Av_j = \sigma_j u_j$.