Math 210A, Fall 2017 (B. Li)

Hints / Sol'ns to HW #8

1\. (1) $\|u+v\|^2 + \|u-v\|^2$

$$= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle$$

$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle + \langle u,u\rangle - \langle u,v\rangle - \langle v,u\rangle + \langle v,v\rangle$$

$$= 2\|u\|^2 + 2\|v\|^2$$

(2) $\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle = \langle u,u\rangle + \langle v,v\rangle = \|u\|^2 + \|v\|^2.$

(3) $|\langle u_n, v_n\rangle - \langle u,v\rangle|$

$$= |\langle u_n, v_n\rangle - \langle u_n, v\rangle + \langle u_n, v\rangle - \langle u,v\rangle|$$

$$\le |\langle u_n, v_n\rangle - \langle u_n, v\rangle| + |\langle u_n, v\rangle - \langle u,v\rangle|$$

$$= |\langle u_n, v_n - v\rangle| + |\langle u_n - u, v\rangle|$$

$$\le \|u_n\| \underbrace{\|v_n - v\|}_{\to 0} + \underbrace{\|u_n - u\|}_{\to 0} \|v\|$$

$$\left.\|u_n\| \le \underbrace{\|u_n - u\|}_{\to 0} + \|u\| \right\} \Rightarrow \{\|u_n\|\} \text{ is bounded.}$$

So, $|\langle u_n, v_n\rangle - \langle u,v\rangle| \to 0.$

(4) Suppose $u_n \to u$. Then

$$\big|\|u_n\| - \|u\|\big| \le \|u_n - u\| \to 0 \Rightarrow \|u_n\| \to \|u\|.$$

$$|\langle u_n, v\rangle - \langle u,v\rangle| = |\langle u_n - u, v\rangle| \le \|u_n - u\|\, \|v\| \to 0 \quad \forall v \in X.$$

Suppose $\|u_n\| \to \|u\|$ and $\langle u_n, v\rangle \to \langle u, v\rangle$ $\forall v \in X$.
Then

$$\|u_n - u\|^2 = \|u_n\|^2 - \langle u_n, u\rangle - \langle u, u_n\rangle + \|u\|^2$$
$$\longrightarrow \|u\|^2 - \langle u, u\rangle - \langle u, u\rangle + \|u\|^2 = 0.$$

$\left(\langle u_n, v\rangle \to \langle u, v\rangle \Longrightarrow \langle v, u_n\rangle \to \langle v, u\rangle. \right)$

2. (1) $a, b \in l^2 \Longrightarrow \sum a_k^2 < \infty \quad \sum b_k^2 < \infty$

$\sum (a_k + b_k)^2 \le 2a_k^2 + 2b_k^2$

$\Longrightarrow \sum (a_k + b_k)^2 < \infty \Longrightarrow a + b \in l^2$

$\lambda a \in l^2$ since $\sum (\lambda a_k)^2 = \lambda^2 \sum a_k^2 < \infty$.

(2) $|a_k b_k| \le \frac{1}{2}(a_k^2 + b_k^2)$

$\Longrightarrow \sum |a_k b_k| \le \frac{1}{2} \sum a_k^2 + \sum b_k^2 < \infty$

$\Longrightarrow \sum a_k b_k$ converges.

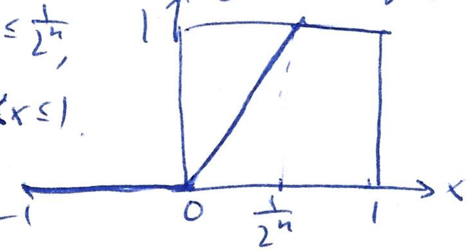
(3) $e^{(k)} = (0, \cdots, 0, \underset{\substack{\uparrow \\ k}}{1}, 0, \cdots)$ $\quad k = 1, 2, \cdots$

$\|e^{(j)} - e^{(k)}\| = \sqrt{2}$.
$\forall j \ne k$.

$\langle e^{(j)}, e^{(k)}\rangle = \delta_{jk}$ So, orthonormal $\Longrightarrow$ linearly indep.

3 (1) No. $f_n(x) = \begin{cases} 2^n x & \text{if } 0 \le x \le \frac{1}{2^n}, \\ 1 & \text{if } \frac{1}{2^n} < x \le 1 \end{cases}$

$a = -1$ $b = 1$.



$f_n \in C([-1, 1])$.

$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$

So, $f_n \to f$ in $L^2(-1, 1)$. f is not cont.

So, $\{f_n\}$ is a Cauchy seq in $C([-1, 1])$ w.r.t. $L^2(-1, 1)$-norm. But the limit is not in $C([-1, 1])$. Note: no cont. function $g \in C([-1, 1])$ will satisfy $\int_{-1}^{1} |f - g|^2 dx = 0$.

(2) $\forall f \in C([a,b])$. Weierstrass $\Rightarrow \exists p_n \in \mathcal{P}$

s.t. $\max_{x \in [a,b]} |p_n(x) - f(x)| \to 0$

Hence $\|p_n - f\|^2_{L^2(a,b)} = \int_a^b |p_n(x) - f(x)|^2 dx$

$$\le \left(\max_{x \in [a,b]} |p_n(x) - f(x)|\right)^2 \underbrace{\int_a^b dx}_{= b-a} \to 0.$$

4. $\langle u_j, u_k \rangle = \langle u_k, u_j \rangle \Rightarrow G$ is symmetric.

Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. $x^T G x = \sum_{j=1}^n \sum_{k=1}^n \langle u_j, u_k \rangle x_j x_k$

$$= \sum_j \sum_k \langle x_j u_j, x_k u_k \rangle$$

$$= \sum_j \langle x_j u_j, \sum_k x_k u_k \rangle$$

$$= \langle \sum_j x_j u_j, \sum_k x_k u_k \rangle = \|\sum_j x_j u_j\|^2 \ge 0$$

If $=0$ then $\sum x_j u_j = 0$.

Suppose G is SPD. then $\forall x \ne 0$. $x^T G x > 0$

$\Longrightarrow \sum x_j u_j \ne 0$. $\Rightarrow u_1, \cdots u_n$ are linearly indep.

Suppose $u_1, \cdots, u_n$ are linearly indep. Then

$\forall x \ne 0$ $\sum x_j u_j \ne 0 \Rightarrow x^T G x = \|\sum_j x_j u_j\|^2 > 0$.

$\Rightarrow G$ is SPD.

~~8~~5. Let u_1, u_2, u_3 be linearly indep. in an inner product space. The Gram-Schmidt orthog. process produces orthogonal vectors v_1, v_2, v_3.

s.t. $\text{span}\{v_1\} = \text{span}\{u_1\}$

$\text{span}\{v_1, v_2\} = \text{span}\{u_1, u_2\}$

$\text{span}\{v_1, v_2, v_3\} = \text{span}\{u_1, u_2, u_3\}$

$v_1 = u_1,$

$$V_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$V_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} V_2.$$

$$Q_0 \cancel{(x)} = p_0 \Longrightarrow Q_0(x) = p_0(x) = \boxed{1}$$

$$Q_1 = P_1 - \frac{\langle p_1, Q_0 \rangle}{\langle Q_0, Q_0 \rangle} Q_0 = x - \frac{\overbrace{\int_{-1}^{1} x\, dx}^{=0}}{\langle Q_0, Q_0 \rangle} Q_0 = \boxed{x}$$

$$Q_2 = p_2 - \frac{\langle p_2, Q_0 \rangle}{\langle Q_0, Q_0 \rangle} Q_0 - \frac{\langle p_2, Q_1 \rangle}{\langle Q_1, Q_1 \rangle} Q_1$$

$$= x^2 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} 1\, dx} \cancel{Q_0} 1 - \frac{\overbrace{\int_{-1}^{1} x^2 \cdot x\, dx}^{=0}}{\langle Q_1, Q_1 \rangle} Q_1$$

$$= \boxed{x^2 - \frac{1}{3}}$$