Math 210 A, Fall 2017 (B. Li)

Hints / Sol'ns to HW # 9

1. $u_n'(x) = n\cos(nx)$
$u_n''(x) = -n^2 \sin(nx)$
$-u_n'' = n^2 \sin(nx) = \lambda_n u_n(x)$.
$u_n(0) = u_n(\pi) = 0$.

$j \neq k$; $j,k \geq 1$: $\int_0^\pi u_j u_k \, dx = \int_0^\pi \sin jx \sin kx \, dx$

$= \int_0^\pi \frac{1}{2}\left[\cos((j-k)x) - \cos((j+k)x)\right] dx$

$= \frac{1}{2}\left[\frac{\sin((j-k)x)}{j-k} - \frac{\cos((j+k)x)}{j+k}\right]\Big|_{x=0}^{\pi} = 0.$

Or $\int_0^\pi u_j u_k \, dx = \int_0^\pi \frac{1}{\lambda_j}(-u_j'')\, u_k \, dx$

since $-u_j'' = \lambda_j u_j$

Integration by parts

$= \frac{1}{\lambda_j}\int_0^\pi u_j' u_k' \, dx + \underbrace{\frac{1}{\lambda_j} u_j' u_k \Big|_{x=0}^{x=\pi}}_{=0}$

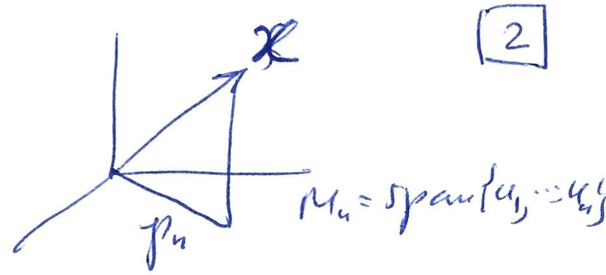
Similarly $\int_0^\pi u_j u_k \, dx = \int_0^\pi u_j \left(-\frac{1}{\lambda_k} u_k''\right) dx$

$= \frac{1}{\lambda_k}\int_0^\pi u_j' u_k' \, dx + \underbrace{\frac{1}{\lambda_k} u_j u_k' \Big|_0^\pi}_{=0}$

$\int_0^\pi u_j u_k \, dx = \frac{1}{\lambda_j}\int_0^\pi u_j' u_k' \, dx = \frac{1}{\lambda_j} \lambda_k \int_0^\pi u_j u_k \, dx$

$\lambda_j \neq \lambda_k \Rightarrow \int_0^\pi u_j u_k \, dx = 0$.

So, $\{u_k\}$ orthogonal!



2. (1) $p_n = \sum_{j=1}^{n} \alpha_j u_j$

$\langle p_n - x, u_k \rangle = 0 \quad k = 1, \cdots, n$

$\Longrightarrow \langle \sum_j \alpha_j u_j, u_k \rangle = \langle x, u_k \rangle$

$\sum_j \alpha_j \langle u_j, u_k \rangle = \langle x, u_k \rangle$

But $\langle u_j, u_k \rangle = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$

$\Longrightarrow \alpha_k = \langle x, u_k \rangle \Longrightarrow p_n = \sum_{k=1}^{n} \langle x, u_k \rangle u_k$

$p_n - x \perp p_n \Longrightarrow x = x - p_n + p_n$

$\|x\|^2 = \|x - p_n\|^2 + \|p_n\|^2$.

$\|p_n\|^2 = \|\sum_k \langle x, u_k \rangle u_k\|^2 = \sum_k \|\langle x, u_k \rangle u_k\|^2$

$= \sum_{k=1}^{n} |\langle x, u_k \rangle|^2 \|u_k\|^2 = \sum_{k=1}^{n} |\langle x, u_k \rangle|^2$.

So, $\|x\|^2 = \|x - p_n\|^2 + \sum_{k=1}^{n} |\langle x, u_k \rangle|^2$

(2) From (1), $\sum_{k=1}^{n} |\langle x, u_k \rangle|^2 \leq \|x\|^2$

Let $n \to \infty$. $\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \leq \|x\|^2$.

3. Proof Let $\delta = \inf_{v \in M} \|v - x\|$

So, $\exists\, u_1, u_2, \cdots \in M$ s.t. $\|u_n - x\| \to \delta$ as $n \to \infty$

$\cancel{\|u_n - u_m\|^2 = 2\|u_n - x\|^2 + 2\|u_m - x\|^2 - 4\|\underbrace{\tfrac{u_n + u_m}{2}}_{\in M \text{ by the convexity}}\|^2}$

$\|u_n - u_m\|^2 = \|(u_n - x) + (u_m - x)\|^2$

$= 2\|u_n - x\|^2 + 2\|u_m - x\|^2 - 4\|\underbrace{\tfrac{u_n + u_m}{2}}_{\in M} - x\|^2$

Since M is convex, $\frac{1}{2}(u_n+u_m) \in M$. Hence

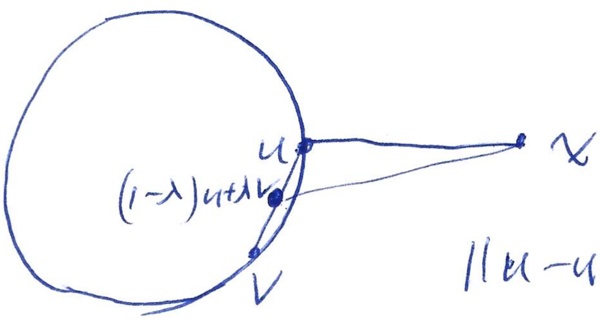
$\|\frac{1}{2}(u_n+u_m)-x\| \geq \delta$.

$$\Longrightarrow \|u_n-u_m\|^2 \leq 2\|u_n-x\|^2+2\|u_m-x\|^2-4\delta^2$$
$$\longrightarrow 2\delta^2+2\delta^2-4\delta^2=0$$

So, $\{u_n\}$ is Cauchy. $u_n \to u$ in X

But M is closed $\Longrightarrow u \in M$.

So, $\left.\begin{array}{l}\|u_n-x\| \to \delta \\ \|u_n-x\| \to \|u-x\|\end{array}\right\} \Longrightarrow \|u-x\|=\delta$
$\leq \|v-x\|$
$\forall v \in M$.



If $\exists u' \in M$ s.t
$\|u'-x\|=\delta$.
then

$$\|u-u'\|^2 = 2\|u-x\|^2+2\|u'-x\|^2 - 4\left\|\frac{u+u'}{2}-x\right\|^2$$
$$\leq 2\delta^2+2\delta^2-4\delta^2=0$$

$\Longrightarrow u=u'$.

Finally, $\forall v \in M$. $\forall \lambda \in (0,1)$

$(1-\lambda)u+\lambda v \in M$ (M is convex)

$$\Longrightarrow \|(1-\lambda)u+\lambda v-x\|^2 > \|u-x\|^2$$
$$\Longrightarrow \underbrace{\|u-x-\lambda(u-v)\|^2}_{\|u-x\|^2+\lambda^2\|u-v\|^2-2\lambda\langle u-x,u-v\rangle} > \|u-x\|^2$$
$$\|u-x\|^2+\lambda^2\|u-v\|^2-2\lambda\langle u-x, u-v\rangle > \|u-x\|^2$$
$$\Longrightarrow \lambda^2\|u-v\|^2-2\lambda\langle u-x,u-v\rangle > 0$$
$$\Longrightarrow \langle u-x, u-v\rangle < \frac{\lambda}{2}\|u-v\|^2$$
$$\langle x-u, v-u\rangle < \frac{\lambda}{2}\|u-v\|^2 \to 0$$
$$\Longrightarrow \langle x-u, v-u\rangle \leq 0.$$

4. (1) $\langle e_k, u\rangle = u_k$ if $u = (u_1, u_2, \dots, u_k, \dots)$.

But $u \in l^2 \Rightarrow \sum u_k^2 < \infty \Rightarrow u_k \to 0$.

So, $\langle e_k, u\rangle \to 0$.

(2) No. If $\exists\, u \in l^2$ s.t. $\|e_k - u\| \to 0$

then $\|e_k - u\|^2 \to 0$.

$$\|e_k - u\|^2 = \|e_k\|^2 + \|u\|^2 - 2\langle e_k, u\rangle$$

$$\longrightarrow 1 + \|u\|^2 - 0 = 1 + \|u\|^2 \neq 0$$

Contradiction!

5. $$\int_0^1 |f_n(x)|^2 dx = \int_0^1 |f(nx)|^2 dx \overset{y=nx}{=\!=} \int_0^n |f(y)|^2 \frac{dy}{n}$$

$$= n \int_0^1 |f(y)|^2 \frac{dy}{n} = \int_0^1 |f(y)|^2 dy = \int_0^1 |f(x)|^2 dx.$$

If $f_n \to g$ in $L^2(0,1)$. then

$$\|f_n - g\|^2 = \underbrace{\|f_n\|^2}_{=1} + \|g\|^2 - 2\underbrace{\langle f_n, g\rangle}_{\to 0} \to 1 + \|g\|^2 \neq 0.$$

Contradiction!

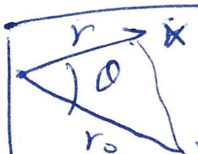
6. If $r < r_0$ then

$$\frac{1}{|\vec{x} - \vec{x}_0|} = \frac{1}{\sqrt{r^2 + r_0^2 - 2 r_0 r \cos\theta}}$$

$$= \frac{1}{r_0\sqrt{1 + \frac{2r}{r_0}\cos\theta + \left(\frac{r}{r_0}\right)^2}}$$

$$= \frac{1}{r_0} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r}{r_0}\right)^n.$$

the case $r > r_0$ is similar!



law of cosine

$\Rightarrow |\vec{x} - \vec{x}_0|^2 = |\vec{x}|^2 + |\vec{x}_0|^2 - 2|\vec{x}||\vec{x}_0| \cdot \cos\theta$.

$\Longrightarrow$

$|\vec{x} - \vec{x}_0| = \sqrt{r^2 + r_0^2 - 2 r r_0 \cos\theta}$