Math 210A: Mathematical Methods in Physical Sciences and Engineering
Fall 2017

Homework Assignment 3 (Due Friday, October 20, 2017)

1. Define $f(x) = 1/x$ if $0 < x \leq 1$ and $f(0) = 0$. For each integer $n \geq 1$, define $f_n(x) = 1/x$ if $1/n \leq x \leq 1$ and $f_n(x) = 0$ if $0 \leq x < 1/n$. Prove the following:
   (1) The sequence of functions $\{f_n\}$ converge to f point-wise on $[0, 1]$;
   (2) The sequence of functions $\{f_n\}$ does not converge in $L^2(0, 1)$.

2. Let $a, b \in \mathbb{R}$ and $a < b$. Denote by $C([a, b])$ the set of all real-valued continuous functions on $[a, b]$.
   (1) Let $f \in C([0, 1])$. Show that
   $$\int_0^1 |f(x)|\, dx \leq \sqrt{\int_0^1 |f(x)|^2 dx}.$$
   (2) Assume that $f_n \in C([0, 1])$ $(n = 1, 2, \dots)$, $f \in C([0, 1])$, and $f_n \to f$ in $L^2([0, 1])$. Prove that $f_n \to f$ in $L^1([0, 1])$.

3. The Heaviside function $H(x)$ is defined by $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$. Let $\phi = \phi(x)$ be a continuously differentiable function on $\mathbb{R}$ that vanishes outside a finite interval $[-A, A]$ for some $A > 0$. Show that
   $$-\int_{-\infty}^{\infty} H(x)\phi'(x)\, dx = \phi(0).$$
   (This shows that, in the sense of distributions, the derivative of Heaviside function is the Dirac δ-function.)

4. A sequence of functions f_n defined on an interval I are said to be uniformly bounded if there exists $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in I$ and all $n \geq 1$. Answer the following questions with justification:
   (1) Assume that g_n is a bounded function on an interval I for each $n \geq 1$ and that $g_n \to g$ on I point-wise. It is true that g is also bounded on I?
   (2) Assume that h_n is a sequence of uniformly bounded functions on an interval I and $h_n \to h$ point-wise on I. It is true that h is also bounded on I?

5. Suppose $u \in C([0, 1])$ satisfy that
   $$\int_0^1 x^n u(x)\, dx = 0 \qquad \forall\, n = 0, 1, \dots.$$
   Prove that $u(x) = 0$ for all $x \in [0, 1]$.