Homework Assignment 6 (Due Wednesday, November 15, 2017)

1. Determine if each of the following statements is true or false and justify your answer:
   (1) If 0 is an eigenvalue of $A$, then $\det A = 0$;
   (2) If $\lambda_1$ and $\lambda_2$ are two distinct eigenvalues of $A$, then $\lambda_1 + \lambda_2$ is also an eigenvalue of $A$;
   (3) If $\lambda$ and $u$ are an eigenvalue and a corresponding eigenvector of $A$, then $\lambda - 7$ and $u$ are an eigenvalue and a corresponding eigenvector of $A - 7I$;
   (4) If $C$ is a real, nonsingular, square matrix, then $C^T C$ is a symmetric positive definite matrix.

2. Let $A$ be a $3 \times 3$ matrix with eigenvalues 0, 1, 2. Find: (1) rank $(A)$; (2) $\det A$; (3) all eigenvalues of $A^T A$; and (4) all eigenvalues of $(A + I)^{-1}$.

3. Let $\lambda_1$ and $\lambda_2$ be the two eigenvalues of a $2 \times 2$ matrix $A$. Suppose the trace of $A$ is 1 and the determinant of $A$ is 2. Calculate $\lambda_1^2 + \lambda_2^2$.

4. Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Find $A^{100}$. (Hint: Diagonalize $A$ to have $T^{-1}AT = D$, i.e., $A = TDT^{-1}$.)

5. Let $P = [p_{ij}]_{n \times n}$ be a stochastic matrix, i.e., $P$ satisfies $0 \leq p_{ij} \leq 1$ for all $i, j$ and $\sum_{j=1}^{n} p_{ij} = 1$ for all $i$. Prove that $\lambda = 1$ is an eigenvalue of $P$ and the vector $e$ with all components being 1 is a corresponding eigenvector.

6. Show that the matrix
   $$
   \begin{bmatrix}
   2 & -1 & -1 & -1 \\
   -1 & 2 & -1 & -1 \\
   -1 & -1 & 2 & -1 \\
   -1 & -1 & -1 & 2 \\
   \end{bmatrix}
   $$
   is symmetric positive definite.

7. Let $f$ be a positive and continuous function on $[0, 1]$. Let $n \geq 1$ be an integer. Define
   $$a_{ij} = \int_{0}^{1} x^{i+j} f(x) \, dx \quad (i, j = 0, 1, \ldots, n).$$
   Show that the $(n + 1) \times (n + 1)$ matrix $A = [a_{ij}]$ is symmetric positive definite.