Math 210A: Mathematical Methods in Physical Sciences and Engineering
Fall 2017

Homework Assignment 6 (Due Wednesday, November 15, 2017)

1. Determine if each of the following statements is true or false and justify your answer:
   (1) If 0 is an eigenvalue of A, then $\det A = 0$;
   (2) If λ_1 and λ_2 are two distinct eigenvalues of A, then $\lambda_1 + \lambda_2$ is also an eigenvalue of A;
   (3) If λ and $\mathbf{u}$ are an eigenvalue and a corresponding eigenvector of A, then $\lambda - 7$ and $\mathbf{u}$ are an eigenvalue and a corresponding eigenvector of $A - 7I$;
   (4) If C is a real, nonsingular, square matrix, then $C^T C$ is a symmetric positive definite matrix.

2. Let A be a 3×3 matrix with eigenvalues $0, 1, 2$. Find: (1) $\operatorname{rank}(A)$; (2) $\det A$; (3) all eigenvalues of $A^T A$; and (4) all eigenvalues of $(A + I)^{-1}$.

3. Let λ_1 and λ_2 be the two eigenvalues of a 2×2 matrix A. Suppose the trace of A is 1 and the determinant of A is 2. Calculate $\lambda_1^2 + \lambda_2^2$.

4. Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Find A^{100}. (Hint: Diagonalize A to have $T^{-1}AT = D$, i.e., $A = TDT^{-1}$.)

5. Let $P = \left[p_{ij}\right]_{n \times n}$ be a stochastic matrix, i.e., P satisfies $0 \le p_{ij} \le 1$ for all i, j and $\sum_{j=1}^{n} p_{ij} = 1$ for all i. Prove that $\lambda = 1$ is an eigenvalue of P and the vector $\mathbf{e}$ with all components being 1 is a corresponding eigenvector.

6. Show that the matrix
$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$
is symmetric positive definite.

7. Let f be a postiive and continuous function on $[0, 1]$. Let $n \ge 1$ be an integer. Define
$$a_{ij} = \int_0^1 x^{i+j} f(x)\, dx \qquad (i, j = 0, 1, \ldots, n).$$
Show that the $(n+1) \times (n+1)$ matrix $A = [a_{ij}]$ is symmetric positive definite.