Math 210A: Mathematical Methods in Physical Sciences and Engineering
Fall 2017

Homework Assignment 9 (Due Friday, December 8)

1. Let $\lambda_n = n^2$ and $u_n(x) = \sin(nx)$ $(n = 1, 2, \dots)$. Verify for each n that $-u_n'' = \lambda_n u_n$ on $(0, \pi)$ and that $u_n(0) = u_n(\pi) = 0$. Is $\{u_n\}_{n=1}^{\infty}$ an orthogonal sequence in $L^2(0, \pi)$?

2. Let $\{u_k\}_{k=1}^{\infty}$ be an orthonormal system in a Hilbert space X. Let $x \in X$.
   (1) Let $n \geq 1$ be any integer. Show that the best approximation of x in the finitely dimensional space $\operatorname{Span}\{u_1, \dots, u_n\}$ is given by $p_n = \sum_{k=1}^{n} \langle x, u_k \rangle u_k$. Moreover, $\|x\|^2 = \|x - p_n\|^2 + \sum_{k=1}^{n} |\langle x, u_k \rangle|^2$.
   (2) Prove Bessel's inequality $\sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2 \leq \|x\|^2$ and further $\lim_{k\to\infty} \langle x, u_k \rangle = 0$.

3. Let X be a real Hilbert space. Let M be a nonempty, closed, convex subset of X. Suppose $x \in X \setminus M$. Prove there exists a unique $u \in M$ such that $\|u - x\| \leq \|v - x\|$ for any $v \in M$. Moreover, u is characterized by $u \in M$ and $\langle x - u, v - u \rangle \leq 0$ for any $v \in M$.

4. For each integer $k \geq 1$, let $e_k \in l^2$ be the vector with the kth component equal to 1 and all others 0.
   (1) Prove that $\{e_k\}$ converges weakly to 0 in l^2, i.e., $\lim_{k\to\infty} \langle e_k, u \rangle = 0$ for any $u \in l^2$.
   (2) Does the sequence $\{e_k\}_{k=1}^{\infty}$ converge to any vector in l^2?

   (Note. This is a typical example that a sequence convergence weekly but not strongly (in norm), due to "oscillations".)

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero, continuous, and 1-periodical function, with 0 average on one period, i.e., $\displaystyle\int_0^1 f(x)\,dx = 0$. Now define $f_n(x) = f(nx)$ $(n = 1, 2, \dots))$. It is known that the inner product in $L^2(0, 1)$ $\langle f_n, g \rangle \to 0$ as $n \to \infty$ for any $g \in L^2(0, 1)$. (You don't need to prove this; though you are encouraged to think about it.) Show that $\displaystyle\int_0^1 |f_n(x)|^2\,dx = \int_0^1 |f(x)|^2\,dx$ for each $n \geq 1$ and that $\{f_n\}_{n=1}^{\infty}$ does not converge to any vector in $L^2(0, 1)$. (Hint: Proof by contradiction.)

6. Suppose there is a point charge Q at $\mathbf{x}_0 \in \mathbb{R}^3$ with $r_0 = |\mathbf{x}_0| > 0$. Then it is known that the electrostatic potential at any point $\mathbf{x} \in \mathbb{R}^3$ different from $\mathbf{x}_0$ is proportional to $1/|\mathbf{x} - \mathbf{x}_0|$. Let $\theta \in [0, \pi]$ be the angle between the two position vectors $\mathbf{x}_0$ and $\mathbf{x}$. Let $r = |\mathbf{x}|$. Show that

$$\frac{1}{|\mathbf{x} - \mathbf{x}_0|} = \begin{cases} \dfrac{1}{r_0} \displaystyle\sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r}{r_0}\right)^n & \text{if } r < r_0, \\ \dfrac{1}{r} \displaystyle\sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{r_0}{r}\right)^n & \text{if } r > r_0. \end{cases}$$

   (Hint: Show first $|\mathbf{x} - \mathbf{x}_0| = \sqrt{r^2 + r_0^2 - 2r_0 r \cos\theta}$. Then use the generating function for Legendre polynomials.)