Math 210B, Winter 2010

Homework Assignment 4

Due Wednesday, March 10, 2010

1. Prove that the Laplacian operator in polar coordinates (r, θ) is given by

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

2. Let k and L be positive constants. Solve the initial-boundary-value problem

$$\begin{aligned} &u_t = k u_{xx} && \text{for } 0 < x < L,\ t > 0,\\ &u(0,t) = 0 \text{ and } u(L,t) = 0 && \text{for } t > 0,\\ &u(x,0) = \begin{cases} 1 & \text{if } 0 < x \le L/2,\\ 2 & \text{if } L/2 < x < L. \end{cases} \end{aligned}$$

3. Let k and L be positive constants. Solve the initial-boundary-value problem

$$\begin{aligned} &u_t = k u_{xx} && \text{for } 0 < x < L,\ t > 0,\\ &u_x(0,t) = 0 \text{ and } u_x(L,t) = 0 && \text{for } t > 0,\\ &u(x,0) = 6 + 4\cos\left(\frac{3\pi x}{L}\right) && \text{for } 0 < x < L. \end{aligned}$$

4. Solve Laplace's equation on $(0,1) \times (0,1)$ with the boundary conditions:

$$u_x(0,y) = 0, \quad u_x(1,y) = 0, \quad u(x,0) = 0, \quad u(x,1) = f(x),$$

where $f(x)$ $(0 \le x \le 1)$ is a known continuous function.

5. Prove the uniqueness of solution to Poisson's equation $\Delta u = f$ in a smooth bounded domain D with the Robin boundary condition $\partial_n u(x) + a(x)u(x) = 0$ $(x \in \partial D)$, where f is a given function on D and $a = a(x)$ is a smooth function such that $a(x) > 0$ for all $x \in \partial D$.

6. Prove Dirichlet's principle for the Neumann boundary condition: among all real-valued functions $v = v(x)$ on a smooth bounded domain D, the value

$$I[v] = \int_D \frac{1}{2}|\nabla v|^2 dV - \int_{\partial D} hv\, dS$$

is smallest for $v = u$, where u is the solution of the Neumann problem

$$\Delta u = 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial n} = h \quad \text{on } \partial D,$$

and h is a given bounded continuous function on ∂D.