Ordinary Differential Equations and Dynamical Systems

Lecture Notes for Math 210 C, Spring 2014

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Plan. Part 1. Linear Systems
Part 2. Nonlinear Systems: local theory
Part 3. Nonlinear Systems: global theory
Part 4. Bifurcation
Part 5. Discrete Dynamical Systems

Part I. Linear Systems

Section 1. Plane Systems. Phase Potraits
Section 2. General Linear Systems.
Appendix. Matrix Exponentials

~~Section I. Plane Systems~~

Some general remarks — introduction

We study
$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1, \cdots, x_n) \\ \vdots \\ \dfrac{dx_n}{dt} = f_n(t, x_1, \cdots, x_n) \end{cases}$$
a system of ODE

with initial conditions

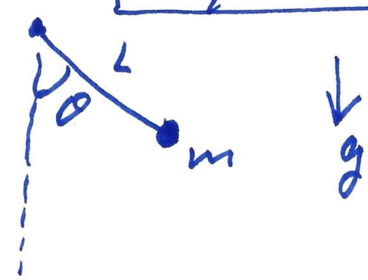
$$x_1(0) = x_{10}, \cdots, x_n(0) = x_{n0}.$$

Here $x_1 = x_1(t)$, $x_2 = x_2(t)$, $\cdots$, $x_n = x_n(t)$ are the unknown functions.

We show how to convert a single high-order eq. into a system of 1st order equations.

<u>Example</u> The motion of a pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$



Let $x_1 = \theta$

$x_2 = \dot{x}_1 \qquad \left(\dot{} = \frac{d}{dt} \right)$

$$\Longrightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{L}\sin x_1 \end{cases}$$

<u>Example</u> If $y = y(t)$ satisfies

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$$

then we can set

$$x_1 = y,\ x_2 = x_1' = y',\ x_3 = x_2' = y'', \cdots, x_n = x_{n-1}' = y^{(n-1)}$$

to get

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \quad \vdots \\ x_{n-1}' = x_n \\ x_n' = F(t, x_1, \cdots, x_{n-1}). \end{cases}$$

Linear Systems (nonhomogeneous if some $b_j \neq 0$)

$$\begin{cases} \dot{x}_1 = a_{11}(t)x_1 + \cdots + a_{1n}(t)x_n + b_1(t) \\ \dot{x}_2 = a_{21}(t)x_1 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x}_n = a_{n1}(t)x_1 + \cdots + a_{nn}(t)x_n + b_n(t) \end{cases}$$

Here all $a_{ij}(t), b_j(t)$ are known functions.

Notation
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad A = [a_{ij}(t)] \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\dot{x} = Ax + b.$$

Strategy: — $b = 0$. A = constant matrix. $n = 2$.
— generalization.

<u>Section 1</u> <u>Plane system</u>: $n = 2$.

Linear, homogeneous plane system with constant coefficients

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad b = 0.$$

$$\dot{x} = Ax. \qquad \begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 \end{cases}$$

Solution Method. $\ddot{x}_1 = a_{11}\dot{x}_1 + a_{12}\dot{x}_2$

from $\dot{x}_1$ equation $= a_{11}\dot{x}_1 + a_{12}(a_{21}x_1 + a_{22}x_2)$

Solve for $a_{22}x_2$ and plug it into $\dot{x}_1 = \cdots$ to get

$$\ddot{x}_1 + p\dot{x}_1 + q x_1 = 0$$

Try $x_1 = e^{\lambda t}$ $\quad \lambda^2 + p\lambda + q = 0 \Rightarrow \lambda_1, \lambda_2.$

(1) λ_1, λ_2: real, distinct $\qquad e^{\lambda_1 t}, e^{\lambda_2 t}$.

(2) $\lambda_1 = \lambda_2$: real. $\qquad e^{\lambda_1 t}, te^{\lambda_1 t}$.

(3) $\lambda_{1,2} = \alpha \pm \beta i \quad \alpha, \beta \in \mathbb{R}$. $\qquad e^{\alpha t}\cos\beta t$, $e^{\alpha t}\sin\beta t$.

Then, Substitute back to $\dot{x}_2 = \cdots$ and solve for x_2. [nonhomogeneous].

Diagonalization Method

Example
$$\begin{cases} \dot{x}_1 = -x_1 - 3x_2 \\ \dot{x}_2 = 2x_2 \end{cases}$$

$A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$ eigenvalues and eigenvectors

$\lambda_1 = -1, \ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2 \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Let $P = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

check: $P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$: diagonal matrix!

Let $y = P^{-1}x \qquad x = Py$

$$\dot{y} = P^{-1}\dot{x} = P^{-1}APy = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} y$$

$$\begin{cases} \dot{y}_1 = -y_1 \\ \dot{y}_2 = 2y_2 \end{cases}$$ Decoupled!

$y_1(t) = c_1 e^{-t}. \qquad y_2(t) = c_2 e^{2t}$

$$x(t) = Py = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} - c_2 e^{2t} \\ c_2 e^{2t} \end{bmatrix}$$

i.e., $$\boxed{\begin{aligned} x_1 &= c_1 e^{-t} - c_2 e^{2t} \\ x_2 &= c_2 e^{2t} \end{aligned}}$$

Alteratively $x_2 = c_2 e^{2t}$

Solve $\dot{x}_1 = -x_1 - 3c_2 e^{2t}$

Nonhomogeneous, 1st-order linear.

Remark. Not every matrix can be diagonalized. But, one can, for 2x2 systems particularly, use the Jordan form

2x2: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

Method of Matrix Exponential

$$x' = Ax \implies x(t) = e^{tA} x(0).$$

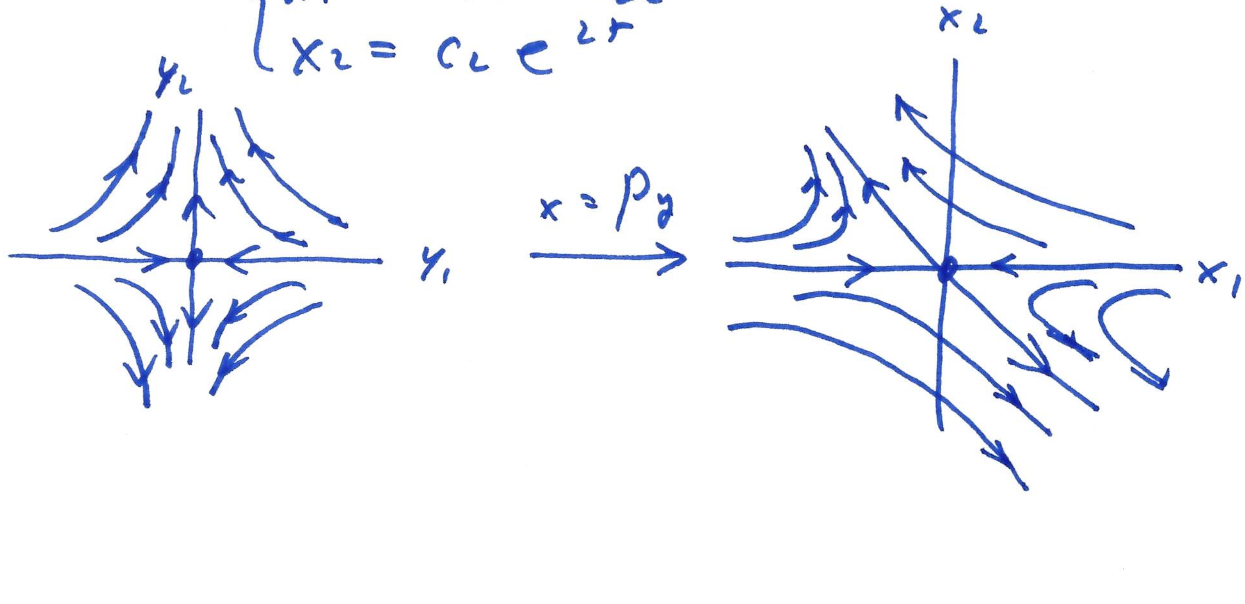
But often e^{tA} is hard to compute. One way is to diagonalize A.

Here we focus on the solution properties: view solution $x = x(t)$ as a trajectory in $\mathbb{R}^n$.

Example (continued) $\quad x' = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} x$.

$$y' = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} y. \qquad \begin{cases} y_1 = c_1 e^{-t} = y_1(0) e^{-t} \\ y_2 = c_2 e^{2t} = y_2(0) e^{2t} \end{cases}$$

$$\begin{cases} x_1 = c_1 e^{-t} - c_2 e^{2t} \\ x_2 = c_2 e^{2t} \end{cases}$$



① $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$: critical point. i.e., $\begin{matrix} x_1(t) \equiv 0 \\ x_2(t) \equiv 0 \end{matrix}$ is an equilibrium solution.

① No trajectories crossing each other: solution is unique. One one trajectory passes through one point.

① Stability of the system around the critical point: going in or leaving away?

Now, consider $x' = Ax$ $\quad x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Let λ_1, λ_2 be the eigenvalues of A:

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$a+d = tr(A)$, $\quad ad-bc = \det A$.

The λ-eq. comes from $\det[\lambda I - A] = 0$.

<u>Case 1</u> $\lambda_1 \neq \lambda_2$: real. <u>Case 2</u> $\lambda_1 = \lambda_2$: real.

<u>Case 3</u> λ_1, λ_2: complex conjugate.

<u>Case 1</u> ~~$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$~~

[After changing coordinates, we have only 3 cases

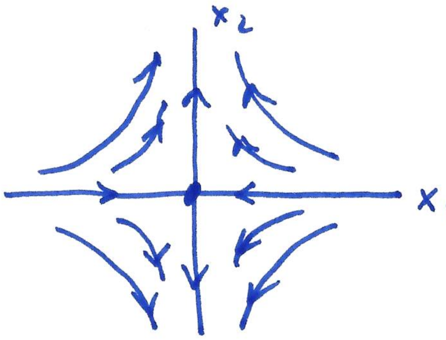
$$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

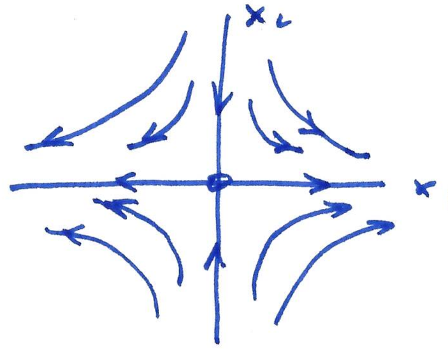
$(\lambda, \mu \in \mathbb{R}.)$ $\qquad \beta \neq 0$.]

Case 1

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \qquad \lambda, \mu \in \mathbb{R}.$$

Case 1.1 $\lambda < 0 < \mu$ or $\lambda > 0 > \mu$





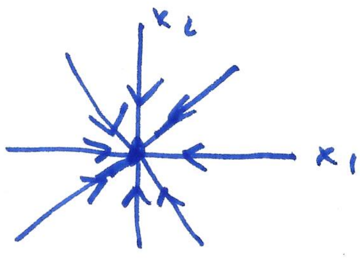
$x_1 = c_1 e^{\lambda t}$
$x_2 = c_2 e^{\mu t}$

$x_1 = c_1 e^{\lambda t}$
$x_2 = c_2 e^{\mu t}$

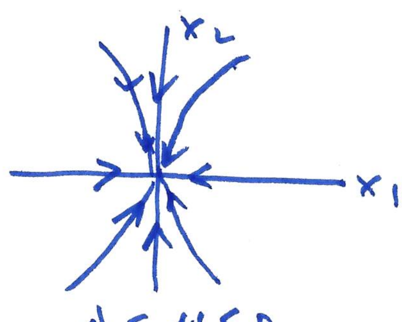
Call $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a saddle node (or saddle point).

Case 1.2 $\lambda \leq \mu < 0$ or $0 > \lambda \geq \mu$.

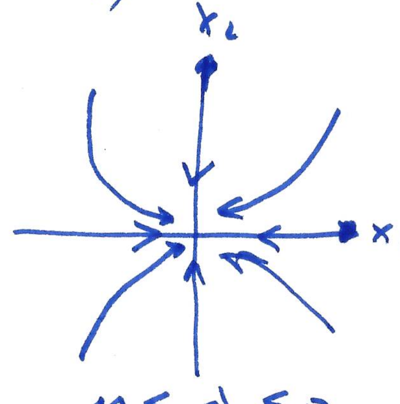
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a stable node (sink)



$\lambda = \mu < 0$



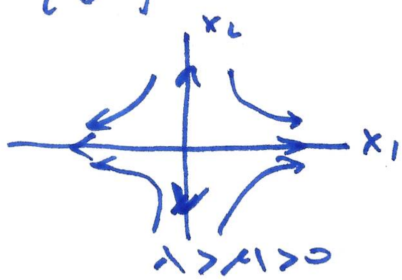
$\lambda < \mu < 0$



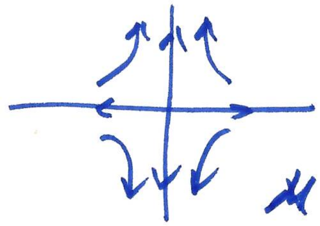
$\mu < \lambda < 0$

Case 1.3 $\lambda > \mu > 0$ or $\mu > \lambda > 0$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an unstable node (source)

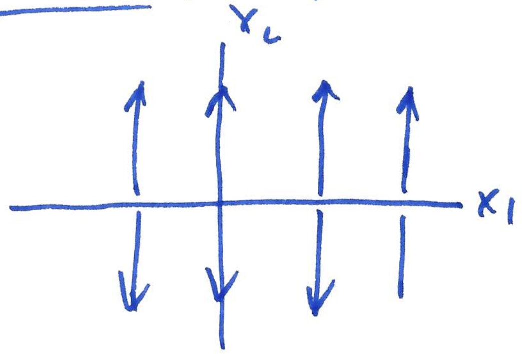


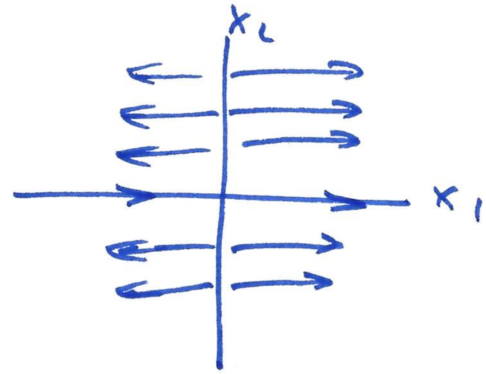
$\lambda > \mu > 0$



$\mu > \lambda > 0$.

Case 1.4. $\lambda = 0 < \mu$ or $\mu = 0 < \lambda$.





$x_1 = x_1(0) \quad \forall t$

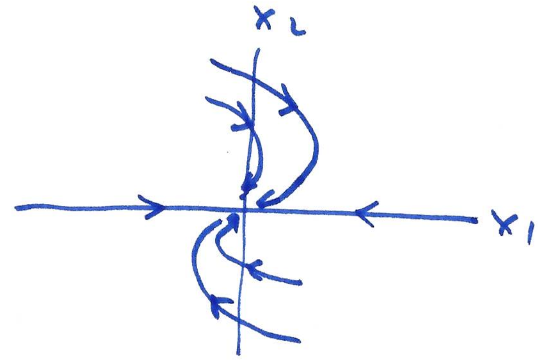
$x_2 = x_2(0) e^{\mu t}$

$\lambda = 0 < \mu$

$\mu = 0 < \lambda$

Case 2 $\quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

Case 2.1 $\quad \lambda < 0$



$\dot{x}_1 = \lambda x_1 + x_2$

$\dot{x}_2 = \lambda \Rightarrow x_2 = x_2(0) e^{\lambda t}$

$\dot{x}_1 - \lambda x_1 = x_2(0) e^{\lambda t}$

$(e^{-\lambda t} x_1)' = x_2(0)$

$e^{-\lambda t} x_1 = x_2(0) t + c_1, \quad x_1 = (x_2(0) t + c_1) e^{\lambda t}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a <u>stable node</u>.

Case 2.2 $\quad \lambda = 0. \quad \dot{x}_1 = x_2 \quad x_1 = c_2 t + c_1$

$\dot{x}_2 = 0 \quad x_2 = c_2$

unstable

Case 2.3 $\quad \lambda > 0$: <u>unstable</u>

Case 3 $\quad B = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. $\quad \alpha, \beta \in \mathbb{R}$, $\beta \neq 0$.

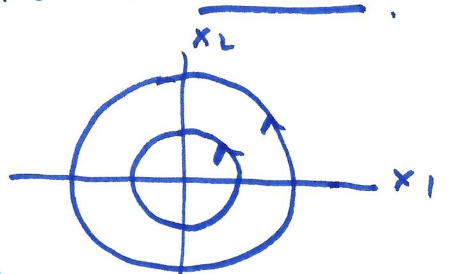
$$x_1(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

$$x_2(t) = c_1 e^{\alpha t} (-\sin \beta t) + c_2 e^{\alpha t} \cos \beta t$$

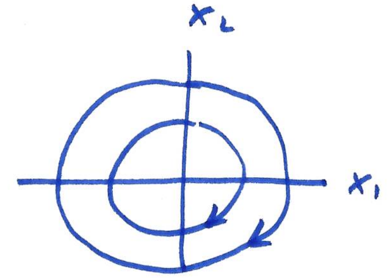
$$x = c_1 e^{\alpha t} \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix} + c_2 e^{\alpha t} \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}$$

Case 3.1 $\quad \alpha = 0$, $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix} + c_2 \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a <u>center</u>.

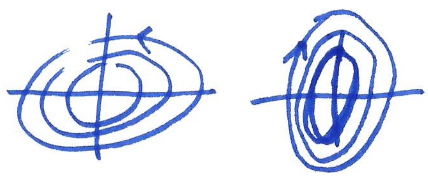


$\beta < 0$

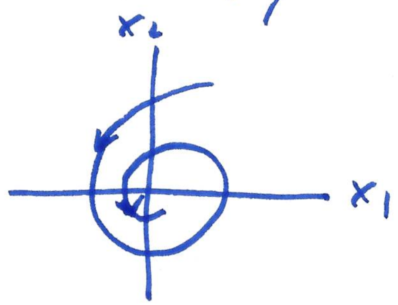


$\beta > 0$

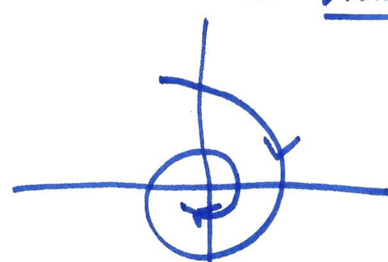
<u>General</u>



Case 3.2 $\quad \alpha < 0$, $\beta \neq 0$. $\quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a <u>stable focus</u> or <u>stable spirals</u>

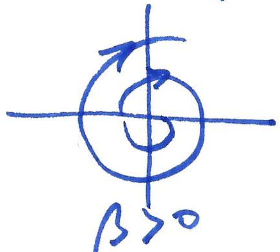


$\beta < 0$

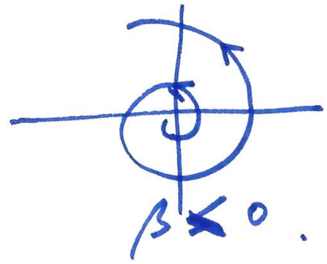


$\beta > 0$

Case 3.3 $\quad \alpha > 0$. $\beta \neq 0$. $\quad$ <u>unstable</u>



$\beta > 0$

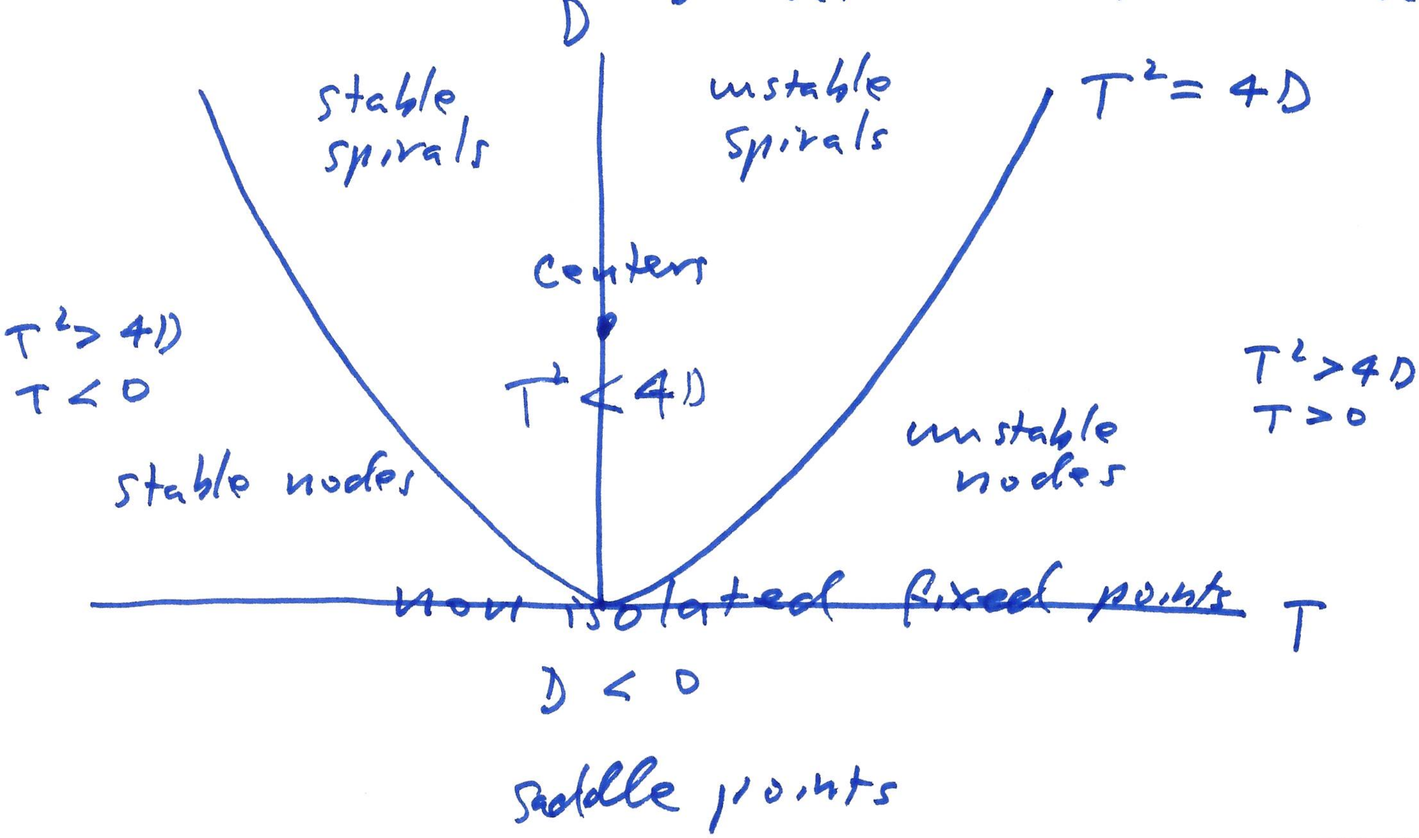


$\beta < 0$.

Summary: The trace-determinant plane

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad T = \text{Trace of } A = a+d$

$D = \text{Determinant of } A = ad - bc$



Analysis $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\quad \det(\lambda I - A) = 0$, $\quad \det \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} = 0$

$(\lambda - a)(\lambda - d) - bc = 0$, $\quad \lambda^2 - (a+d)\lambda \underbrace{-bc}_{+ad} = 0$

$$\boxed{\lambda^2 - T\lambda + D = 0}$$

$\lambda_{1,2} = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$.

$T^2 - 4D > 0$: real, distinct

$T^2 - 4D = 0$: real, repeated

$T^2 - 4D < 0$: complex (nonzero imaginary) $\left\{ \begin{array}{l} T < 0: \text{ spiral sink} \\ T > 0: \text{ spiral source} \\ T = 0: \text{ center} \end{array} \right.$

Section 1.2 General ~~Linearized~~ Linear Systems

$\dot{x} = Ax$ A: $n \times n$ constant matrix

$$x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Solution $\quad x = e^{At} x(0)$

$$e^{At} = e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

Fundamental Theorem for Linear Systems (with constant coefficients) The initial-value problem $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ has a unique solution $x(t) = e^{At} x_0$.

Proof If $x(t) = e^{At} x_0$ then $x(0) = e^0 x_0 = I x_0 = x_0$.

and $x' = A e^{At} x_0 = Ax$.

If $y(t)$ is also a solution, then $z(t) = e^{-At} y(t)$ satisfies

$$z'(t) = -A e^{-At} y(t) + e^{-At} y'$$
$$= -A e^{-At} y(t) + e^{-At} A y(t) = 0$$

since $A e^{-At} = e^{-At} A$.

$z(t) \equiv z(0) = e^0 y(0) = I y(0) = x_0$

So, $y(t) = (e^{-At})^{-1} z(t) = e^{At} z(0)$
$= e^{At} x_0 = x(t)$. $\forall t$. Q.E.D.

Next, we study solutions as trajectories, study how stable the origin (as a fixed pt - critical point) is. High-dim problems are difficult. But, we can study some examples first.

If A is diagonalizable then the solution can be obtained.

e.g., $A = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$. $\quad x' = Ax$

$$\Longrightarrow \quad x = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}.$$

If all $\lambda_j < 0$ then 0 is a <u>stable node</u> (a sink)

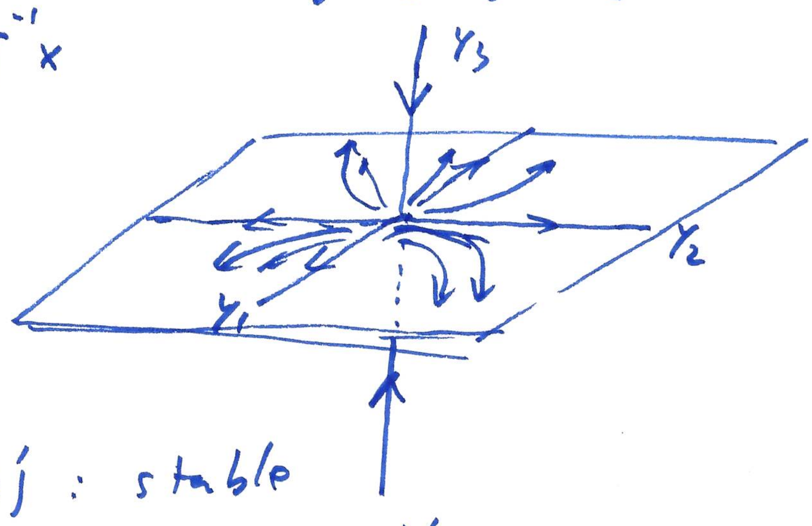
If all $\lambda_j > 0$ then 0 is an <u>unstable node</u> (a source)

If some $\lambda_j > 0$ others < 0. then a <u>saddle</u>

<u>Example</u> $\quad \dot{x} = \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{bmatrix}}_{A} x \qquad [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$

$$T^{-1}AT = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad T = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$Y = T^{-1}x$



span$\{v_3\}$: stable

span$\{v_1, v_2\}$: unstable.

if A is diagonalizable.

In general. $\dot{y} = Dy = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} y$

$$T = [\underbrace{\vec{v}_1 \cdots \vec{v}_n}_{\text{linearly indep.}}] \qquad T^{-1}AT = D.$$

$$y(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

$$x = Ty = [\vec{v}_1 \cdots \vec{v}_n] \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

$$= \sum_{j=1}^{n} c_j e^{\lambda_j t} \vec{v}_j .$$

If, for instance, $\lambda_1, \dots, \lambda_k > 0$, $\lambda_{k+1} < 0, \dots, \lambda_n < 0$

then $\forall x_0 \in \operatorname{span}\{\vec{v}_{k+1} \cdots \vec{v}_n\}$ then

$$x_0 = \sum_{j=k+1}^{n} c_j \vec{v}_j \quad \text{for some } c_j: \ j = k+1, \dots, n.$$

then $$x(t) = \sum_{j=k+1}^{n} c_j e^{\lambda_j t} \vec{v}_j$$

Solves (uniquely): $\begin{cases} x' = Ax \\ x(0) = x_0 \in \operatorname{span}\{v_{k+1}, \dots, v_n\} \end{cases}$.

Since $\lambda_j < 0$ $(j = k+1, \dots, n)$. $x(t) \to 0$ exponentially.

Similarly, $x_0 \in \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ then

$$x(t) = \sum_{j=1}^{k} \hat{c}_j e^{\lambda_j t} v_j \qquad \left(x(0) = x_0 = \sum_{1}^{k} \hat{c}_j v_j \right)$$

Since each $\lambda_j > 0$ $(j = 1, \dots, k)$ then

$x(t) \to \infty$ $\left(e^{\lambda_j t} \text{ dominants with } \lambda_j = \max_{1 \le i \le k} \lambda_i \right)$

[componentwise]

Now. consider the case that some eigenvalues are real (distinct) and some are ~~&~~ complex conjugate.

$\dot{x} = Ax$. Assume eigenvalues of A are

$$\underbrace{\lambda_1, \cdots, \lambda_k}_{\text{real, distinct}} \quad \alpha_{1} \pm \beta_{1} i, \cdots, \alpha_m \pm \beta_m i$$

$\mu_1^{\pm}, \cdots, \mu_m^{\pm}$: different

$\mu_j^{\pm} = \alpha_j \pm \beta_j i, \ \beta_j \neq 0$.

$$k + \underbrace{2\cancel{(m-k)}}_{2m} = n. \qquad 2m - k = n.$$

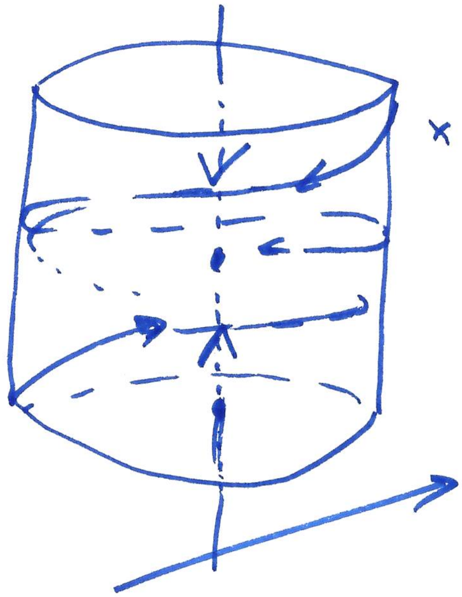
$$x = Ty \qquad \dot{y} = Dy.$$

$$T^{-1}AT = D = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & B_1 & & \\ & 0 & & & \ddots & \\ & & & & & B_m \end{bmatrix}$$

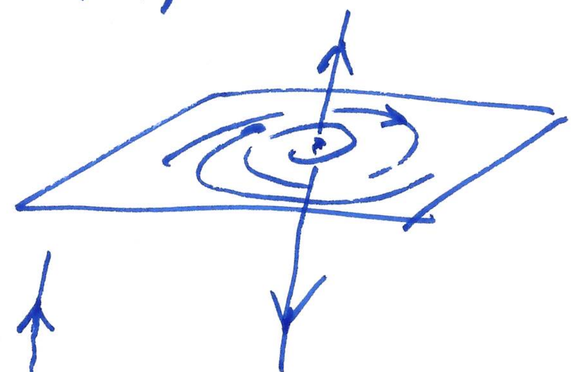
$$B_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix} \quad (j = 1, \cdots, m)$$

$$y(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_k e^{\lambda_k t} \\ (a_1 \cos\beta_1 t + b_1 \sin\beta_1 t) e^{\alpha_1 t} \\ (-a_1 \sin\beta_1 t + b_1 \cos\beta_1 t) e^{\alpha_1 t} \\ \vdots \\ (a_m \cos\beta_m t + b_m \sin\beta_m t) e^{\alpha_m t} \\ (-a_m \sin\beta_m t + b_m \cos\beta_m t) e^{\alpha_m t} \end{bmatrix}$$

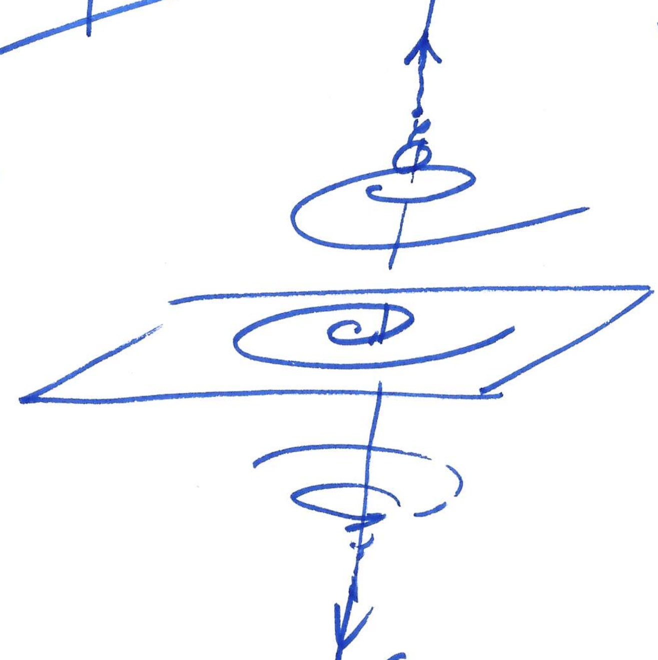
Example $x' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x$



a spiral center



Examples



spiral, saddle

Now, complex eigenvalues only with real parts $= 0$

— harmonic oscillators

Example $x = \begin{bmatrix} \cancel{x_1} \\ \cancel{x_2} \\ \cancel{x_3} \\ \cancel{x_4} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{bmatrix}$. $\begin{cases} x_j' = y_j \\ y_j' = -\omega_j^2 x_j \end{cases}$ $j = 1, 2$.

$$T^{-1}AT = \begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} x_1(t) \\ y_1(t) \\ x_2(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \cos\omega_1 t + b_1 \sin\omega_1 t \\ -a_1\omega_1 \sin\omega_1 t + b_1 \omega_1 \cos\omega_1 t \\ a_2 \cos\omega_2 t + b_2 \sin\omega_2 t \\ -a_2\omega_2 \sin\omega_2 t + b_2\omega_2 \cos\omega_2 t \end{bmatrix}.$$

$x_1(t), y_1(t)$: periodic sol. $\frac{2\pi}{\omega_1}$

$x_2(t), y_2(t)$: - - - - - - - - $\frac{2\pi}{\omega_2}$

The solution $y(t) = \begin{bmatrix} \vdots \end{bmatrix}$ is periodic $\Longleftrightarrow$

$\exists\ m, n \in \mathbb{Z}$ s.t. $\frac{m}{\omega_1} = \frac{n}{\omega_2}$.

In this the period is $\tau = \frac{2\pi m}{\omega_1} = \frac{2\pi n}{\omega_2}$.

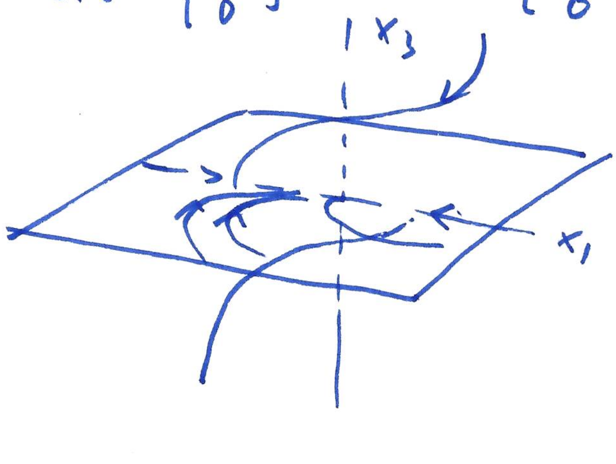
Polar coordinates. $(x_j, y_j) \to (r_j, \theta_j)$

$$\begin{cases} r_j' = 0 \\ \theta_j' = -\omega_j \end{cases} \qquad j = 1, 2.$$

Finally, consider repeated (real) eigenvalues.

<u>Example</u> $x' = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} x$

$$x = c_1 e^{\lambda t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{\lambda t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{\lambda t}\begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix}$$



$\lambda < 0$.

General solution formula using Jordan forms

Theorem Let A be a real matrix with real eigenvalues $\lambda_1, \cdots, \lambda_k$ and complex eigenvalues $\lambda_{k+1} = a_{k+1} + i b_{k+1}$, $\bar{\lambda}_{k+1} = a_{k+1} - i b_{k+1}$, $\cdots$, $\lambda_n = a_n + i b_n$. $\bar{\lambda}_n = a_n - i b_n$ (all $a_j, b_j \in \mathbb{R}$, $j = k+1, \cdots, n$). Then exists a basis $\{u_1, \cdots, u_k, v_{k+1}, w_{k+1}, \cdots, v_n, w_n\}$ for $\mathbb{R}^{2n-k}$ such that $P = [u_1, \cdots, u_k, v_{k+1}\ w_{k+1}, \cdots\ v_n\ w_n]$ is invertible and

$$P^{-1}AP = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{bmatrix},$$

where each of the elementary Jordan blocks $B_1, \cdots, B_r$ is either of the form

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \left(= \lambda I + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix} \right) \qquad (1)$$

for λ one of the real eigenvalues of A or of the form

$$\begin{bmatrix} D & I_2 & 0 & \cdots & 0 \\ 0 & D & I_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & D & I_2 \\ 0 & 0 & \cdots & 0 & D \end{bmatrix} \qquad (2)$$

with $D = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, for $\lambda = a + bi$ one of the complex eigenvalues of A.

Theorem Under the assumption and conclusion of the previous Thm, the initial-value problem

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

has the unique solution

$$x(t) = P \begin{bmatrix} e^{B_1 t} & & \\ & \ddots & \\ & & e^{B_r t} \end{bmatrix} P^{-1} x_0.$$

If B_j is an $m\times m$ matrix of the form (1) then

$$e^{B_j t} = e^{\lambda t}\begin{bmatrix} 1 & t & t^2/2! & \cdots & t^{m-1}/(m-1)! \\ 0 & 1 & t & \cdots & t^{m-2}/(m-2)! \\ 0 & 0 & 1 & \cdots & t^{m-3}/(m-3)! \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

If B_j is a $2m\times 2m$ matrix of the form (2) then

$$e^{B_j t} = e^{at}\begin{bmatrix} R & Rt & Rt^2/2! & \cdots & Rt^{m-1}/(m-1)! \\ 0 & R & Rt & \cdots & Rt^{m-2}/(m-2)! \\ 0 & 0 & R & \cdots & Rt^{m-3}/(m-3)! \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & Rt \\ 0 & 0 & 0 & \cdots & R \end{bmatrix},$$

where $R = \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$ (rotation by bt).

In particular, each component of $x(t)$ is a linear combination of

$$t^k e^{\alpha t}\cos\beta t,\quad t^k e^{\alpha t}\sin\beta t$$

where $\alpha+\beta i$ is an eigenvalue of A, and $0\le k\le n_A-1$. n_A is the number of rows (and columns) of A.

Note that the vectors that form P are generalized eigenvectors of A: ~~If~~ u is a generalized eigenvector corresponding to an eigenvalue λ of A of multiplicity m if $\exists k$: $(A-\lambda I)^k u = 0$, $u\neq 0$, $1\le k\le \cancel{N}\,m$, ~~where A is $N\times N$~~.

In general, a matrix may not be diagonalizable: there may not be enough linearly independent eigenvectors. But, there are just enough linearly independent generalized eigenvectors to make A in a ~~the~~ Jordan form.

Stability Theory for $\dot{x} = Ax$.

More precisely, how stable is the critical pt 0?
Assume A is $n \times n$, real. Denote

E^s = span of generalized eigenvectors of A corresponding to eigenvalues with negative real parts. — call it stable subspace for $\dot{x} = Ax$.

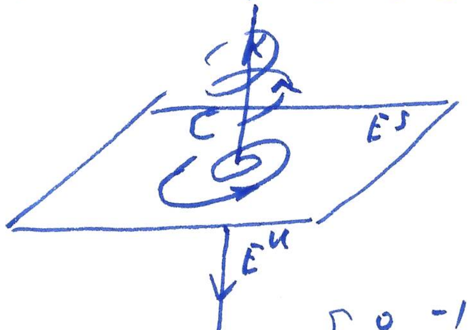
E^c = span of generalized eigenvectors of A corresponding to eigenvalues with zero real parts — center subspace for $\dot{x} = Ax$,

E^u = span of generalized eigenvalues of A corresponding to eigenvalues with positive real parts. — unstable subspace for $\dot{x} = Ax$.

[s: stable, c: center, u: unstable]

Example $A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad \lambda_1 = -2+i, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 3 \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



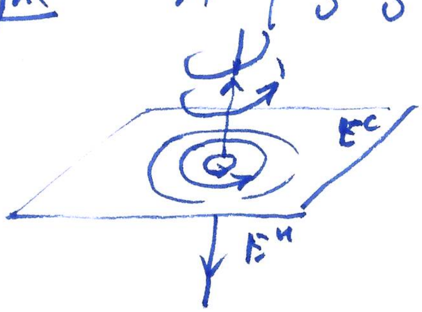
$E^s = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$E^u = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

] difference $\lambda_1 = -2+i$ or $\lambda_1 = i$.

Example $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad \lambda_1 = i \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 2 \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



$E^c = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$E^u = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Theorem Given A, $n\times n$, real. we have

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c,$$

and each of E^s, E^u, E^c is invariant under the flow $\cancel{x}e^{At}$. i.e. $x_0 \in E^s$ (or E^u or E^c) then $x(t) = e^{At}x_0 \in E^s$ (or E^u or E^c).

Theorem The following are equivalent [A: $n\times n$, real]

(a) All eigenvalues of A have negative real part.

(b) $\forall x_0 \in \mathbb{R}^n$: $e^{At}x_0 \to 0$ $(t\to\infty)$

$\forall x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, $e^{At} \to \infty$ (as $t \to -\infty$)

(c) $\exists$ $\underbrace{a, c, m, M, k}_{>0}$ s.t. $\forall x_0 \in \mathbb{R}^n$, $t \in \mathbb{R}$

$$m|t|^k e^{-at}|x_0| \le |e^{At}x_0| \le M e^{-ct}|x_0|.$$

A similar result is true for the case that all eigenvalues of A have positive real part.

A final remark: nonhomogeneous system

$$\begin{cases} \dot{x} = Ax + b(t) \\ x(0) = x_0 \end{cases} \qquad b(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix} \qquad A: n\times n$$

Solution $x(t) = e^{At}x_0 + e^{At}\int_0^t e^{-A\tau} b(\tau)\,d\tau.$

Appendix Matrix Exponential

Definition $e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k = I + B + \frac{1}{2!}B^2 + \cdots$ (B: a square matrix)

This converges absolutely!

Properties

① $e^0 = I$ [Obvious]

② $e^{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}} = \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}$

③ $(e^B)^T = e^{B^T}$ $\quad \left(\sum_k \frac{1}{k!} B^k\right)^T = \sum_k \frac{1}{k!} (B^k)^T = \cdots$

④ e^B is always invertible.

$(e^B)^{-1} = e^{-B}$. [since $e^B \cdot e^{-B} = e^0 = I$]

⑤ If A is diagonalizable: $P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

then $e^A = P \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} P^{-1}$

$$\Big[\; \cancel{P^{-1}} \left(\sum_k \frac{1}{k!} A^k\right) \cancel{P} = \sum \frac{1}{k!} \left(P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}\right)^k$$

$$= \sum \frac{1}{k!} \underbrace{P[\;\;]P^{-1}}\;\underbrace{P[\;\;]P^{-1}} \cdots \underbrace{P[\;\;]P^{-1}}$$

$$= \sum \frac{1}{k!} P[\cdots]^k P^{-1} = P\left(\sum \frac{1}{k!}[\;\;]\right)P^{-1}$$

$$= P \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} P^{-1} \; \Big].$$

Example

$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$

⑥ $AB = BA \Rightarrow e^A e^B = e^B e^A = e^{A+B}$.

Pf $\quad e^A e^B = \left(\sum \frac{1}{k!} A^k\right)\left(\sum \frac{1}{k!} B^k\right) = \sum C_k$

$$C_k = \frac{1}{k!}A^k + \frac{1}{(k-1)!}A^{k-1}B + \frac{1}{(k-2)!}A^{k-2}\frac{1}{2!}B^2 + \cdots + A\frac{1}{(k-1)!}B^{k-1} + \frac{1}{k!}B^k$$

$$= \frac{1}{k!}\left[A^k + \binom{k}{k-1}A^{k-1}B + \cdots + \binom{k}{1}AB^{k-1} + B^k\right]$$

Use $AB=BA$

$$\overset{\downarrow}{=} \frac{1}{k!}(A+B)^k. \quad \text{Q.E.D.}$$

⑦ If $A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{bmatrix}$ then $e^A = \begin{bmatrix} e^{A_1} & & \\ & \ddots & \\ & & e^{A_r} \end{bmatrix}$

where $A_1, \dots, A_r$ are square matrices.

⑧ $\det e^A = e^{\operatorname{Tr} A} > 0$.

Pf. Jordan form: $P^{-1} A P = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$

$A = PJP^{-1}$, $e^A = P e^J P^{-1}$, $e^J = \begin{bmatrix} e^{J_1} & & \\ & \ddots & \\ & & e^{J_s} \end{bmatrix}$

$\det e^A = \det e^J = \prod_{j=1}^{s} \det e^{J_j}$.

Each $J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}_{m_k \times m_k}$ $\operatorname{Tr} J_k = \lambda_k + \dots + \lambda_k = m_k J_k$

$$\operatorname{Tr} A = \operatorname{Tr} J = \sum_{k=1}^{s} \operatorname{Tr} J_k = \sum_{k=1}^{s} m_k J_k$$

$$\det e^{J_k} = \underbrace{e^{\lambda_k} \cdots e^{\lambda_k}}_{m_k} = e^{m_k \lambda_k} = e^{\operatorname{tr} J_k}$$

$$\det e^A = e^{\sum_k \operatorname{tr} J_k} = e^{\operatorname{Tr} A}$$ Q.E.D.

⑨ $\vec{u}$ is an eigenvector of A corresponding eigenvalue λ $\Rightarrow$ $\vec{u}$ is e-vector of e^A for e^λ.

Theorem If A is $n \times n$, then $e^A = \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$ for some scalars $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$. (dep. on A)

Pf By the Cayley-Hamilton Thm: $f(A) = 0$ where $f(\lambda) = \det(\lambda I - A) = \hat{\alpha}_0 + \hat{\alpha}_1 \lambda + \dots + \hat{\alpha}_n \lambda^n$. $\hat{\alpha}_n = 1$

Thus. A^n = linear combination of $I, A, \dots, A^{n-1}$.

A^{n+1} = linear combination of $A, A^2, \dots, A^n$

= linear combination of $I, A, \dots, A^{n-1}$.

$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$ is a linear combination of $I, A, \dots, A^{n-1}$. Q.E.D.

Thm. If $J = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{m\times m}$ is a Jordan block ($\lambda \in \mathbb{C}$).

then
$$e^J = e^\lambda \begin{bmatrix} 1 & 1 & 1/2! & \cdots & 1/(m-1)! \\ 0 & 1 & 1 & \cdots & 1/(m-2)! \\ 0 & 0 & 1 & \cdots & 1/(m-3)! \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

(see p.17.).
Try $m=3$. and calculate directly to see the pattern.

Theorem If A is an $n\times n$ nonsingular matrix then there exists an $n\times n$ matrix B such that $e^B = A$.

Now, look at e^{At}. We have

① $\frac{d}{dt} e^{At} = Ae^{At}$

② $e^{At}e^{As} = e^{A(t+s)}$

③ $e^{At}\cdot e^{-At} = I. \Longrightarrow e^{-At} = (e^{At})^{-1}.$