Part 2. Nonlinear Systems: Local Theory

Section 1. Existence/Uniqueness of Solutions
Solution Dependence on data/parameters

Section 2. Equilibrium/Critical Points.
Linearization. Stability Theory

Section 3 Gradient and Hamiltonian Systems

Section 1 Solution Existence, Uniqueness, and Dependence on Data

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

$$\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases} \qquad f: E \to \mathbb{R}^n, \quad E \subset \mathbb{R}^n \text{: open} \qquad x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad x_0 \in E.$$

$x' = f(x)$: called an autonomous system (in contrast to $f(x,t)$, for instance).

$x = x(t)$. $\dot{x} = f(x)$ means $\dot{x}(t) = f(x(t))$.

For nonlinear systems, solutions may not exist, or may have many solutions, or.... Bad!

Example $\begin{cases} x' = 3x^{2/3} \\ x(0) = 0 \end{cases}$ $n = 1$.

Two solutions: $u(t) = t^3$, $v(t) = 0$.

Example $\begin{cases} \dot{x} = x^2 \\ x(0)=1 \end{cases}$ (quadratic nonlinearity)

Unique solution: $x(t) = \frac{1}{1-t}$, $t\in(-\infty,1)$. Sol'n is only defined ~~up to~~ in $(-\infty,1)$.

$$\lim_{t\to 1^-} x(t) = \infty.$$

Some preparation

f is continuously differentiable in E means all the components of f have continuous first-order partial derivatives all in E.

$$C^1(E) = \left\{ f: E \to \mathbb{R}^n \;\middle|\; f=\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix},\ \text{all } \frac{\partial f_i}{\partial x_j} \text{ exist and are continuous on } E \right\}$$

$$C^1(E) = C^1(E,\mathbb{R}^n) = [C^1(E)]^n$$

$Df(x_0)$ is a linear mapping from $\mathbb{R}^n \to \mathbb{R}^n$.

$x\in\mathbb{R}^n$: $\quad Df(x_0)x = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x_0)\, x_j \in \mathbb{R}^n$

Identify $Df(x_0)$ as a matrix:

$$Df(x_0) = \left[\frac{\partial f_i(x_0)}{\partial x_j} \right]_{n\times n}.$$

So, $$Df(x_0)x = \begin{bmatrix} \vdots \\ \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(x_0)\, x_j \\ \vdots \end{bmatrix}$$

$f\in C^2(E)$: all $\frac{\partial^2 f_i}{\partial x_j \partial x_k}$ exist and are continuous.

For each $x_0 \in E$, $D^2 f(x_0)$ (if exists) is a bilinear mapping: $D^2 f(x_0): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$$D^2 f(x_0)(y, z) = \sum_{j,k=1}^{n} \frac{\partial^2 f(x_0)}{\partial x_j \partial x_k} y_j \cdot z_k \in \mathbb{R}^n$$

for any $y, z \in \mathbb{R}^n$.

$f: E \to \mathbb{R}^n$ is said to be Lipschitz continuous, if there exists a constant L (depending on E, f, n) such that

$$|f(x) - f(y)| \le L\,|x - y| \qquad \forall x, y \in E.$$

Here $|\cdot|$ means L^2-norm. e.g., $a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$

$|a| = \|a\| = \sqrt{1^2 + 2^2 + 3^2} = ?$

If $f \in C^1(E, \mathbb{R}^n)$ then f is locally Lipschitz: $|f(x) - f(y)| = |Df(\xi)(x-y)| \le L\,|x-y|$.

The Fundamental Existence-Uniqueness Theorem

Suppose $E \subset \mathbb{R}^n$ is open, $f \in C^1(E, \mathbb{R}^n)$, $x_0 \in E$. Then $\exists\, a > 0$ such that $\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$ has a unique sol'n for $t \in [-a, a]$.

(Note: May chane 0 to t_0, $[-a, a]$ to $[t_0 - a, t_0 + a]$.)

Sketch of Proof

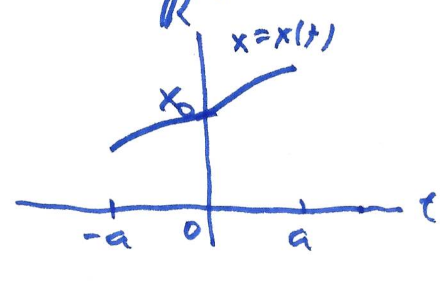
Step 1. f is locally Lipschitz at x_0: $\exists\, K > 0$, $\exists\, N_\varepsilon(x_0)$ $(= B(x_0, \varepsilon))$ (a neighborhood — just a ball of x_0.) s.t.

$$|f(x) - f(y)| \le K\,|x - y|, \quad \forall x, y \in N_\varepsilon(x_0).$$

For a smaller $N_{\varepsilon/2}(x_0) = N(x_0) = N_0$, denote

$$M \equiv \max_{x \in N_0} |f(x)|$$

$$b = \frac{\varepsilon}{2}.$$



<u>Step 2</u> (Picard Iteration). Define

$$u_0(t) = x_0$$

$$(*) \qquad u_{k+1}(t) = x_0 + \int_0^t f(u_k(s))\,ds$$

[The idea is that if u is a solution to $\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$ then $u(t) = x_0 + \int_0^t f(u(s))\,ds$.]

$a < 1/K$

u_k is cont. on $[-a,a]$ welled and

$0 < a \le b/M$ (choose a so small)

$$|u_{k+1}(t) - x_0| \le \left|\int_0^t f(u_k(s))\,ds\right| \le Ma.$$

for $-a \le t \le a$:

<u>Step 3</u> $\quad |u_{j+1}(t) - u_j(t)| \le \int_0^t |f(u_j(s)) - f(u_{j-1}(s))|\,ds$

$$\le K \int_0^t |u_j(s) - u_{j-1}(s)|\,ds$$

$$\le Ka \max_{[-a,a]} |u_j(t) - u_{j-1}(t)|$$

$j = 1$: $\quad |u_1(t) - u_0(t)| = |u_1(t) - x_0| = \left|\int_0^t f(u_0(s))\,ds\right|$

$$\le |t|\,|f(x_0)| \le aM \le b$$

$j = 2$: $\quad |u_2(t) - u_1(t)| \le Kab$

Induction $\quad |u_{j+1}(t) - u_j(t)| \le (Ka)^j b.$

$$\|u_m - u_k\| = \max_{[-a,a]} |u_m(t) - u_k(t)| \to 0 \text{ as } m, k \to \infty$$

Since $\quad |u_m(t) - u_k(t)| = \sum_{j=k}^{m} |u_{j+1}(t) - u_j(t)|$

$$\le \sum_{j=k}^{\infty} |u_{j+1}(t) - u_j(t)|$$

$$\le \sum_{j=k}^{\infty} (Ka)^j b = \frac{(Ka)^k}{1 - Ka}\, b \to 0 \text{ as } \begin{matrix} k \to \infty \\ m \to \infty \end{matrix}.$$

Choose a so that $Ka < 1$.

<u>Step 4</u> $\quad u(t) = \lim_{m \to \infty} u_m(t)$ exists.

Let $k \to \infty$ in $(*)$ $\quad \left.\begin{matrix} u(t) = x_0 + \int_0^t f(u(s))\,ds \\ u(0) = x_0. \end{matrix}\right\} \Rightarrow$ existence.

Step 5 Uniqueness. The difference $w = u - v$ of two solutions u, v satisfies $w(0) = 0$ and

$$w(t) = \int_0^t f(u(s)) - f(v(s))\, ds$$

$$\|w\| = \max_{[-a,a]} |w(t)| = \left| \int_0^{t_{max}} w(s)\, ds \right|$$

for some $t_{max} \in [-a, a]$

$$\le \int_0^{|t_{max}|} |f(u(s)) - f(v(s))|\, ds$$

$$\le K \int_0^a |u(s) - v(s)|\, ds$$

$$\le Ka \|w\|$$

But $Ka < 1$. So $\|w\| = 0$. Q.E.D.

We now consider how solutions depend on initial data.

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \qquad \begin{cases} \dot{y} = f(y) \\ y(0) = y_0 \end{cases}$$

$x = x(t)$, $y = y(t)$, $t \in [-a, a]$

Local existence theorem $\Rightarrow$ existence of x, y.

Question If $|y_0 - x_0| \ll 1$, does $|y(t) - x(t)| \ll 1$?

Integral equations: $x(t) = x_0 + \int_0^t f(x(s))\, ds$
$y(t) = y_0 + \int_0^t f(y(s))\, ds$

$$y(t) - x(t) = y_0 - x_0 + \int_0^t [f(y(s)) - f(x(s))]\, ds$$

$$|y(t) - x(t)| \le |y_0 - x_0| + \int_0^t |f(y(s)) - f(x(s))|\, ds \quad (t > 0)$$

$$\le |y_0 - x_0| + K \int_0^t |y(s) - x(s)|\, ds$$

Lip. condition

By Gronwall's inequality (see below), for
$g(t) = |y(t) - x(t)|$, we have

$$g(t) \le g(0) + K\int_0^t g(s)\,ds$$

$$\Longrightarrow \quad g(t) \le g(0)e^{Kt}.$$

For $t \in [0, T]$ with $T > 0$ fixed.
$g(0) = |y_0 - x_0| \to 0$ (i.e., $y_0 \to x_0$)
$\Longrightarrow \quad g(t) = |y(t) - x(t)| \longrightarrow 0$
This means that $x = x(t)$: $\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$ depends on x_0 continuously.

<u>Gronwall's Inequality</u> Suppose $g(t)$ is a continuous real-valued function such that $g(t) \ge 0$ and

$$g(t) \le C + K\int_0^t g(s)\,ds \qquad \forall t \in [0, T]$$

for some $T > 0$, where C, K are positive constants.
Then $\quad g(t) \le Ce^{Kt} \quad \forall t \in [0, T]$.

<u>Proof</u> Let $G(t) = C + K\int_0^t g(s)\,ds \quad (t \in [0, T])$
Then $G(t) \ge g(t)$ and $G(t) > 0 \quad \forall t \in [0, T]$.
$G'(t) = Kg(t)$. So,

$$\frac{G'(t)}{G(t)} = \frac{Kg(t)}{G(t)} \le K \qquad \forall t \in [0, T]$$

$$\frac{d}{dt}\log G(t) \le K$$

$\int_0^t$:

$$\log G(t) \le Kt + \underbrace{\log G(0)}_{=\log C}$$

$$G(t) \le G(0)e^{Kt} = Ce^{Kt}.$$

But $g(t) \le G(t)$. <u>Q.E.D.</u>

Theorem (Dependence on Initial Conditions) Let E be an open subset of $\mathbb{R}^n$, $x_0 \in E$, $f \in C^1(E; \mathbb{R}^n)$. Then $\exists\, a>0,\ \delta>0$ such that $\forall y \in B(x_0, \delta)$, the initial-value problem

The ball in $\mathbb{R}^n$ centered at x_0 with radius δ

$$\begin{cases} \dot{x} = f(x) \\ x(0) = y \end{cases}$$

has a unique solution $u = u(t, y)$ with $u \in C^1([-a,a] \times B(x_0, \delta))$. Moreover, for each y, $u(t,y)$ is twice continuously differentiable of $t \in [-a, a]$.

We will not give a full, detailed proof. We just make some remarks.

(1) $\dot{u}(t,y) = f(u(t,y))$

Chain Rule.

But $f \in C^1$. So, $\ddot{u}(t,y) = Df(u(t,y))\,\dot{u}(t,y)$

That is why $u(t,y)$ is twice continuously differentiable on t.

(2) Why $u = u(t,y)$ is continuously differentiable on y?

$$\begin{cases} \partial_t u(t,y) = f(u(t,y)) \\ u(0,y) = y \end{cases}$$

Integral representation:

$$u(t,y) = y + \int_0^t f(u(s,y))\,ds$$

Formally, $\underbrace{\partial_y u(t,y)}_{\text{a matrix}} = \underset{\substack{\uparrow \\ \text{Identity matrix}}}{I} + \int_0^t \underbrace{Df(u(s,y))}_{\text{matrix}} \underbrace{\partial_y u(s,y)}_{\text{matrix}}\,ds$

<u>Theorem</u> (<u>Dependence on Parameters</u>) Let E be an open set in $\mathbb{R}^{n+m}$, $x_0 \in \mathbb{R}^n$, $\mu_0 \in \mathbb{R}^m$, $(x_0, \mu_0) \in E$. Let $f \in C^1(E; \mathbb{R}^n)\cancel{m)}$. Then $\exists\, a > 0$, and $\delta > 0$ such that $\forall\, y \in B(x_0, \delta)$ $\forall\, \mu \in B(\mu_0, \delta)$ the initial-value prob.

$$\begin{cases} \dot{x} = f(x, \mu) \\ x(0) = y \end{cases}$$

has a unique solution $u = u(t, y, \mu)$ with $u \in C^1([-a, a] \times B(x_0, \delta) \times B(\mu_0, \delta))$.

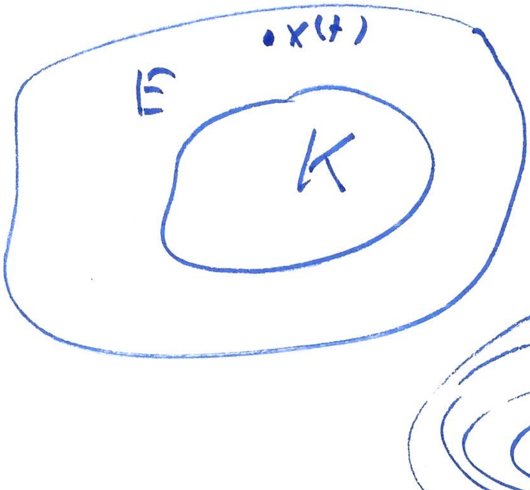
Finally in this section, we discuss the maximal interval of existence. That is, we extend $[-a, a]$ of solution $x = x(t)$ to largest possible interval of t on which the solution exists. The existence of such an interval can be proved.

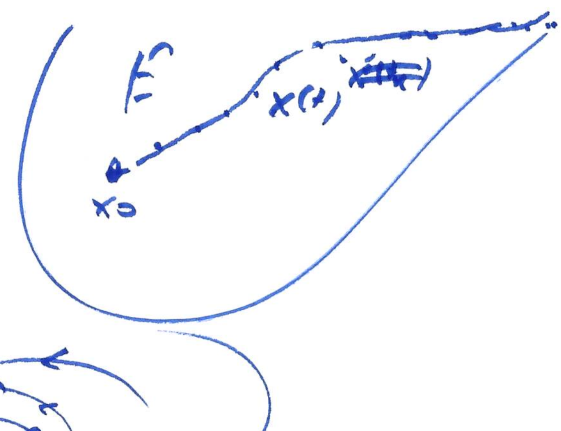
<u>Theorem</u> Let $E \subseteq \mathbb{R}^n$ be open, $x_0 \in E$, and $f \in C^1(E; \mathbb{R}^n)$. Let (α, β) be the maximal interval of solution $x = x(t)$ to

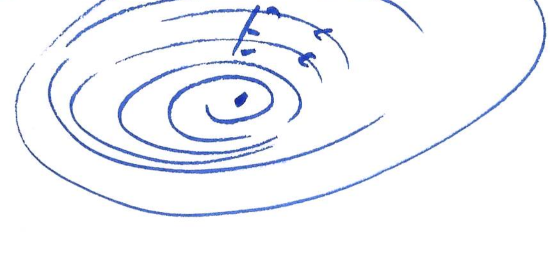
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

Then if $K \subset E$ is compact then $\exists\, t \in (\alpha, \beta)$ s.t. $x(t) \notin K$.

In particular, if $\lim_{t \to \beta^-} x(t)$ exists, then it is on the boundary of E.







Section 2 Stability Theory

Linearization
Liapunov
Topological

Let E be an open set in $\mathbb{R}^n$, $x_0 \in E$, and $f \in C^1(E, \mathbb{R}^n)$.
We consider $\dot{x} = f(x)$.
If $f(x_0) = 0$ then $x(t) \equiv x_0$ is a solution, called an equilibrium sol'n or stationary solution. In this case, call x_0 a critical point or fixed pt. or equilibrium point of the dynamical system $\dot{x} = f(x)$, or of the mapping $f: E \to \mathbb{R}^n$.

How stable is an equilibrium sol'n $x(t) \equiv x_0$?
For a linear system $\dot{x} = Ax$. $x_0 = 0$ is an equilibrium.
Its stability is determined by eigenvalues of A.

Linearization Let x_0 be a critical point of f.

$$\dot{x} = f(x) = \underbrace{f(x_0)}_{=0} + Df(x_0)(x - x_0) + O((x-x_0)^2)$$

Let $h = x - x_0$ $(|h| \ll 1)$. $\dot{x} = (x - x_0)' = \dot{h}$

So,
$$\dot{h} = Df(x_0)\, h + O(h^2)$$

$[A = Df(x_0)$ $n \times n$ matrix.$]$

$$h(t) = h(0)\, e^{tDf(x_0) + O(h)}$$
$$= h(0)\, e^{tDf(x_0)}(1 + O(h))$$
$$= h(0)\, e^{tDf(x_0)} + O(h)$$

So, for very small $t > 0$, $x(t) = h(t) + x_0$ is determined mainly by $A = Df(x_0)$.

Call $Df(x_0)$ the linearization of $\dot{x} = f(x)$ at a critical pt x_0. Or: $\dot{h} = Df(x_0)\, h$ the linearization of $\dot{x} = f(x)$ at x_0.

Definition. Let x_0 be a critical point of $\dot{x} = f(x)$. $A = Df(x_0)$.

(1) x_0 is a sink if all eigenvalues of A have negative real part.

(2) .... source .... positive .....

(3) .... saddle if all .... nonzero ....,
and at least one eigenvalue has a positive real part
and at least one eigenvalue has negative real part.

(4) x_0 is a hyperbolic if all eigenvalues of A have nonzero real part.

For convenience, we often assume $x_0 = 0$. Otherwise consider $y = x - x_0$. $\dot{y} = \dot{x} = f(x) = f(y + x_0) = \hat{f}(y)$.
$[\, y(0) = x(0) - x_0 = 0 \,]$

We shall also write the sol'n $x = x(t)$ to $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$
as $x = x(t, x_0)$ or $x = \phi_t(x_0)$.

The Stable Manifold Theorem Let E be open in $\mathbb{R}^n$ and $f \in C^1(E, \mathbb{R}^n)$, Let $0 \in E$ be a critical point of f. (i.e., $f(0) = 0$), Let ϕ_t be the flow by $\dot{x} = f(x)$. Suppose $Df(x_0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then (1) there exists a k-dimensional differential manifold S tangent to the stable subspace E^s of $\dot{x} = Df(0)x$ at 0, such that $\phi_t(S) \subseteq S$ $\forall t \geq 0$ and $\lim_{t \to \infty} \phi_t(x_0) = 0$ $\forall x_0 \in S$.
(2) There exists an $n-k$ dimensional differentiable manifold U tangent to the unstable subspace E^u of $\dot{x} = Df(0)x$ at 0 such that $\phi_t(U) \subseteq U$ and

$$\lim_{t\to-\infty}\phi_t(x_0)=0 \quad \forall x_0\in U. \quad (t\to-\infty \text{ not } t\to+\infty)$$

<u>**Example**</u>
$$\begin{cases}\dot x_1 = -x_1\\ \dot x_2 = -x_2+x_1^2\\ \dot x_3 = x_3+x_1^2\end{cases} \qquad A=Df(0)=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$

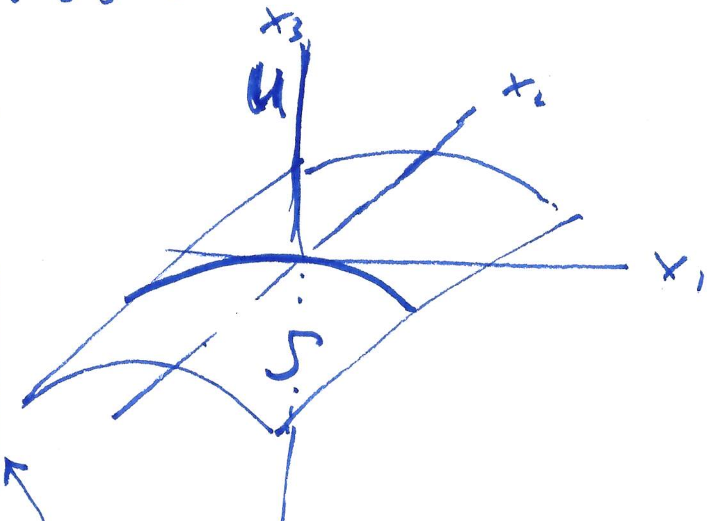
Details. $f(x)=\begin{bmatrix}-x_1\\-x_2+x_1^2\\x_3+x_1^2\end{bmatrix}$ $\quad Df(x)=\begin{bmatrix}-1&0&0\\2x_1&-1&0\\2x_1&0&1\end{bmatrix}$

$Df(0)=A$.

For the linearized system $\dot h = Ah$.

$$E^s=\operatorname{span}\left\{\begin{bmatrix}-1\\0\\0\end{bmatrix},\begin{bmatrix}0\\-1\\0\end{bmatrix}\right\}=\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

$$E^u=\operatorname{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}.$$



We can solve $\dot x = f(x)$:

$$\begin{cases}x_1(t)=c_1e^{-t}\\ x_2(t)=c_2e^{-t}+c_1^2(e^{-t}-e^{-2t})\\ x_3(t)=c_3e^{t}+\dfrac{c_1^2}{3}(e^{t}-e^{-2t})\end{cases}$$

where $c=x(0)$.

$$\phi_t(x_0)=\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}=\phi_t(c).$$

$$\lim_{t\to+\infty}\phi_t(c)=0 \iff c_3+\frac{c_1^2}{3}=0.$$

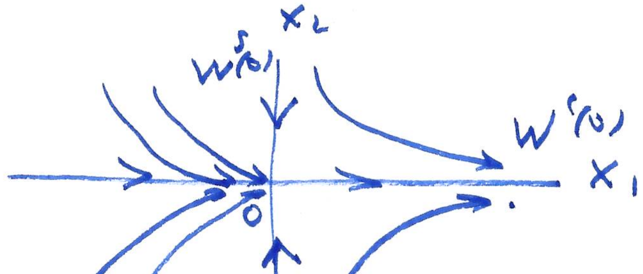
So. $S=\left\{c\in\mathbb{R}^3:\ c_3+\dfrac{c_1^2}{3}=0\right\}$ ← This is tangent to x_1x_2-plane which is E^s for $\dot x = Ax$, $A=Df(0)$.

$$\lim_{t\to-\infty}\phi_t(c)=0 \iff c_1=c_2=0$$

$$U=\{c\in\mathbb{R}^3:\ c_1=c_2=0\}$$

<u>The Center Manifold Theorem</u> Let $E \subseteq \mathbb{R}^n$ be open and $f \in C^r(E;\mathbb{R}^n)$ for some $r \geq 1$. ~~Let~~ Suppose $0 \in E$, and $f(0)=0$. Suppose $A = Df(0)$ has k eigenvalues with negative real part, j eigenvalues with positive real part, and $m = n-k-j$ eigenvalues with zero real part. Then, there exists an m-dimensional center manifold $W^c(0)$ of class C^r tangent to the center space E^c of $\dot{x} = Df(0)x$ at 0; there exists a k-dimensional stable manifold $W^s(0)$ of class C^r tangent to the stable subspace of $\dot{x} = Df(0)x$ at 0; and there exists a j-dimensional unstable manifold $W^u(0)$ of class C^r tangent to the unstable subspace E^u of $\dot{x} = Df(0)x$ at 0. Furthermore, $W^c(0)$, $W^s(0)$, and $W^u(0)$ are invariant under the flow ϕ_t of $\dot{x} = f(x)$.

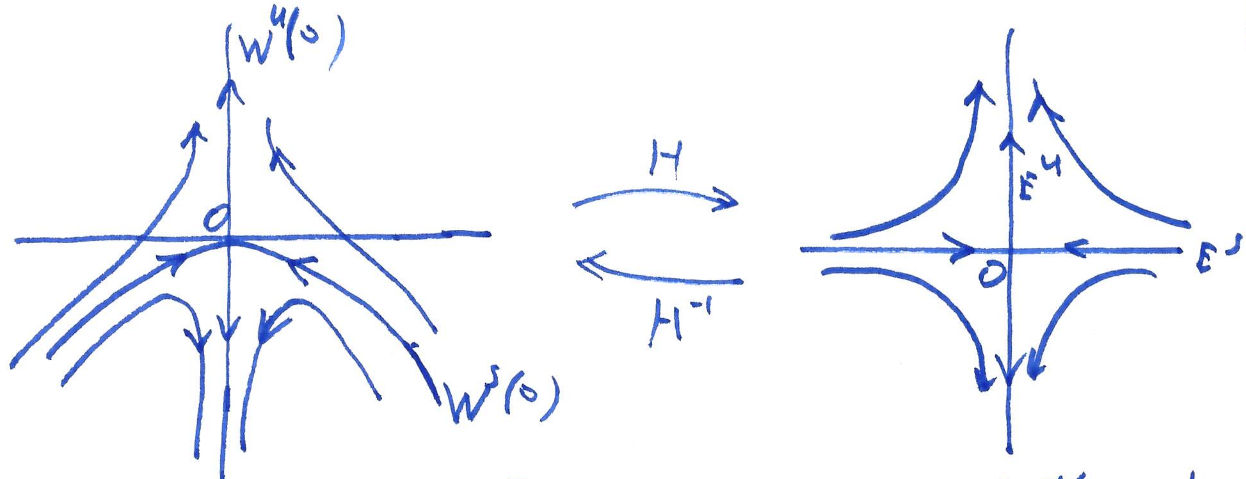
<u>Example</u> $\begin{cases} \dot{x}_1 = x_1^2 \\ \dot{x}_2 = -x_2 \end{cases}$



<u>The Hartman-Grobman Thm</u> If $0 \in E$ is a hyperbolic critical point of $f \in C^1(E,\mathbb{R}^n)$, then $\exists\, U$ open in $\mathbb{R}^n$, $0 \in U$, V open in $\mathbb{R}^n$, $0 \in \mathbb{R}^n$, $H: U \to V$ homeomorphism (1-1, onto, continuous) such that $\forall x_0 \in U$ $\exists$ open interval ~~I_0~~ $I(x_0) \subseteq \mathbb{R}$ with $0 \in I(x_0)$ s.t.

$$H \circ \phi_t(x_0) = e^{Df(0)t} H(x_0), \quad \forall x_0 \in U,\ \forall t \in I_0.$$

[Solution local structure: similar to linear system]



<u>Method of Liapunov</u> (1892, ~~Ph.~~ doctoral thesis)

So far, all results are about non-hyperbolic equilibria.

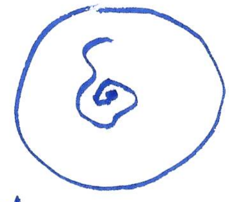
$\dot{x} = f(x)$ determines flow $\phi_t(x_0)$: $x = \phi_t(x_0)$ solves

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

$\phi_t(x) =$ "particle location at t if it started at x."

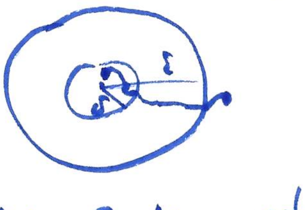
<u>Definition</u> Let ϕ_t be the flow of $\dot{x} = f(x)$, and $x_0 \in E$ a critical point of $f \in C^1(E, \mathbb{R}^n)$. ($E \subseteq \mathbb{R}^n$: open).

① x_0 is <u>stable</u>, if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall x \in B(x_0, \delta)$
$\phi_t(x) \in B(x_0, \varepsilon)$ $\forall t \geq 0$

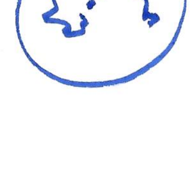


② x_0 is <u>unstable</u> if not stable.
i.e., $\exists \varepsilon_0 > 0$, $\forall \delta > 0$, $\exists x_\delta \in B(x_0, \delta)$
s.t. $\phi_t(x_\delta) \notin B(x_0, \varepsilon_0)$ for some $t \geq 0$

$\underbrace{x(t)}_{=\phi_t(x_\delta)}$ is outside $B(x_0, \varepsilon_0)$ for some t ($>>1$).



③ x_0 is ~~asym~~ asymptotically stable if it is stable and $\exists \delta > 0$
s.t. $x \in B(x_0, \delta) \Longrightarrow \lim_{t \to \infty} \phi_t(x) = x_0$



Remarks

① Asymptotically stable = stable + limit.
limit alone does not imply x_0 is stable.

② For $n=2$: stable node / focus $\overset{(\not\Leftarrow)}{\Rightarrow}$ asymptotically stable

unstable node / focus or saddle $\Rightarrow$ unstable

center $\Rightarrow$ stable but not asymptotically stable

③ In general, any hyperbolic equilibrium is either asymptotically stable or unstable.

④ In general, if x_0 is stable equilibrium then no eigenvalues of $Df(x_0)$ have positive real part.

⑤ Thm: If x_0 is a sink, and $Re(\lambda_j) < -\alpha < 0$ $(j=1,\cdots,n)$ for all eigenvalues λ_j of $Df(x_0)$. then $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $\forall x\in B(x_0,\delta)$,

$$|\phi_t(x) - x_0| \le \varepsilon e^{-\alpha t} \quad \forall t \ge 0.$$

[This is a corollary of the Stable Manifold Thm.]

Let $f\in C^1(E,\mathbb{R}^n)$. consider $\dot{x} = f(x)$ which defines the flow ~~$x(t) =$~~ $\phi_t(x)$. Let $V\in C^1(E, \cancel{\mathbb{R}^n})$. Denote
scalar.

$$\dot{V}(x) = \frac{d}{dt} V(\phi_t(x))\Big|_{t=0}$$

We have $\dot{V}(x) = DV(x)\cdot \dot{\phi}_t(x) = \underbrace{DV(x)}_{\text{vector}}\cdot\underbrace{f(x)}_{\text{vector}}$.

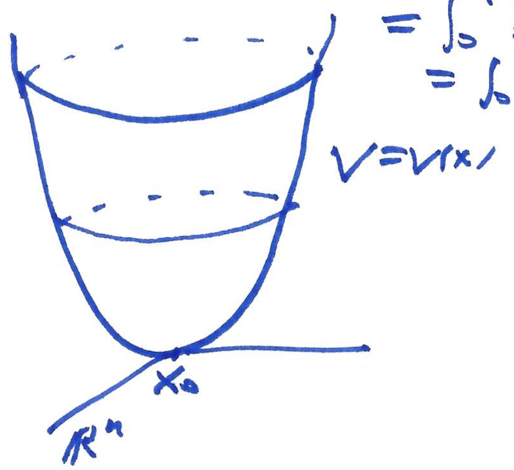
chain rule

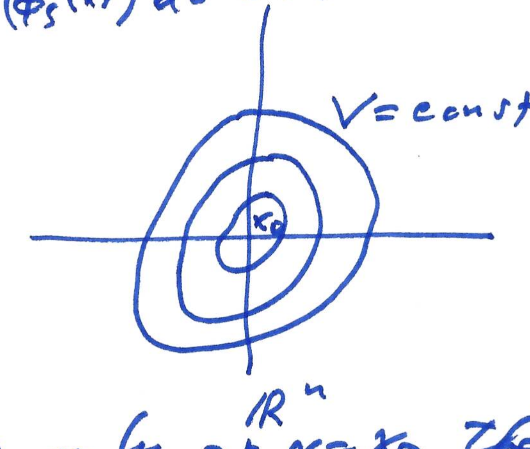
Important note:
If $\dot{V}(x) \le 0$ $\forall x\in E$ then $V(\phi_t(x)) \le V(x)$ $\forall x \in E$, $\forall t\ge 0$.

Proof $V(\phi_t(x)) - V(x) = V(\phi_t(x)) - V(\phi_0(x)) = \int_0^t \frac{d}{ds}[V(\phi_s(x))]\,ds$

$$= \int_0^t DV(\phi_s(x))\cdot \frac{d}{ds}\phi_s(x)\,ds = \int_0^t DV(\phi_s(x))\cdot f(\phi_s(x))\,ds$$

$$= \int_0^t \dot V(\phi_s(x))\,ds \le 0. \quad \text{Q.E.D.}$$





If $\dot V(x) \le 0$ with $\dot V(x) = 0$ only at $x = x_0$, then x_0 is stable.

Assume $f(x_0) = 0$

Definition Let $E \subseteq \mathbb{R}^n$ be open, $x_0 \in E$, and $f \in C^1(E)$. A function $V : E \to \mathbb{R}$ is called a *Liapunov function*, if ① $V \in C^1(E)$, ② $V(x_0) = 0$, $V(x) > 0$ if $x \ne x_0$, and ③ $\dot V(x) \le 0$ $\forall x \in E$. $V : E \to \mathbb{R}$ is a *strict Liapunov function* if ①, ② and $\dot V(x) < 0$ $\forall x \in E$, $x \ne x_0$.

Note: V depends on E, x_0, and f.

Theorem. Let E, x_0, f be as above.

① If $\exists$ a Liapunov function $V \in C^1(E)$ at x_0 then x_0 is stable.

② If $\exists$ a strict Liapunov function $V \in C^1(E)$ at x_0 then x_0 is asymptotically stable.

③ If $\exists$ $V \in C^1(E)$ such that $\dot V(x) > 0$ $\forall x \in E$, $x \ne x_0$, then x_0 is unstable. $V(x) > 0$ $\forall x \ne x_0$, $V(x_0) = 0$.

Proof (skip it in class) W.l.o.g. assume $x_0 = 0 \in E$.

① Let $\varepsilon > 0$ be such that $\overline{B(0,\varepsilon)} \subset E$. Let

$$m_\varepsilon = \min_{x \in S_\varepsilon} V(x) > 0, \text{ where } S_\varepsilon = \partial B(0,\varepsilon) = \{x \in \mathbb{R}^n : |x| = \varepsilon\}.$$

Since V is continuous and $V(0)=0<V(x)$ $(x\neq 0)$, $\exists\,\delta>0$ s.t. $|x|<\delta \Rightarrow V(x)<m_\varepsilon$. Since $\dot V(x)\le 0$ $(x\in E)$, $V(x(t))$ decreases if $x=x(t)$ satisfies $\dot x=f(x)$, i.e., $V(x)$ decreases along trajectories of $\dot x=f(x)$. Thus, if $\phi_t(x)$ is the flow of $\dot x=f(x)$, it follows that $\forall x_0\in B(0,\delta)$, $t\ge 0$. $V(\phi_t(x_0))\le V(x_0)<m_\varepsilon$. Now suppose for $|x_0|<\delta$ $\exists\, t_1>0$ s.t. $|\phi_{t_1}(x_0)|=\varepsilon$, i.e. $\phi_{t_1}(x_0)\in S_\varepsilon$. Then, since $m_\varepsilon=\min\limits_{S_\varepsilon} V$, we would have $V(\phi_{t_1}(x_0))\ge m_\varepsilon$, contradicting the above inequality. Thus, $|x_0|<\delta$, $t\ge 0$ $\Rightarrow |\phi_t(x_0)|<\varepsilon$. So, 0 is a stable equilibrium point.

② By part ①: $|x_0|<\delta \Rightarrow \phi_t(x_0)\in B(0,\varepsilon)$ $\forall t\ge 0$. Let $t_k\uparrow\infty$. Since $\overline{B(0,\varepsilon)}$ is compact, for any $\exists$ subseq of t_k, not relabeled, s.t. $\phi_{t_k}(x_0)\to y_0\in\overline{B(0,\varepsilon)}$. We show $y_0=0$ later. Thus any seq. $\{\phi_{t_k}(x_0)\}$ has a further subseq converging to 0. Thus $\phi_t(x_0)\to 0$ as $t\to\infty$.

Now, suppose $\phi_{t_k}(x_0)\to y_0$. Since $V(x)$ is strictly decreasing along trajectories of $\dot x=f(x)$ and $V(\phi_{t_k}(x_0))\to V(y_0)$, we must have $V(\phi_t(x_0))>V(y_0)$ $\forall t>0$. If $y_0\neq 0$, then for some $s>0$, $V(\phi_s(y_0))<V(y_0)$. By the continuity $V(\phi_s(y))<V(y_0)$ if y is close to y_0. But then for $y=\phi_{t_k}(x_0)$ $(k\gg 1)$

$$V(\phi_s(y))=V(\phi_s(\phi_{t_k}(x_0)))=V(\phi_{s+t_k}(x_0))<V(y_0).$$

A contradiction. Thus $y_0=0$.

③ Let $M=\max\limits_{\overline{B(0,\varepsilon)}} V$. Since $\dot V(x)>0 \Rightarrow V(x)$ increases strictly on trajectories of $\dot x=f(x)$. Thus, $\forall\delta>0$, $\forall x_0\in B(0,\delta)\setminus\{0\}$, $V(\phi_t(x_0))>V(x_0)>0$, $\forall t>0$. Hence

$\inf_{t \geq 0} \dot{V}(\phi_t(x_0)) = m > 0$. Thus $V(\phi_t(x_0)) - V(x_0) \geq mt$ for all $t \geq 0$. Thus $V(\phi_t(x_0)) > mt > M$ if $t >> 1$, i.e., $\phi_t(x_0)$ lies outside $\overline{B(0, \varepsilon)}$. Hence, 0 is unstable.

Q.E.D.

Example

$$\dot{x}_1 = -x_2^3$$
$$\dot{x}_2 = x_1^3.$$

$x_0 = 0$: nonhyperbolic equilibrium point

$$V(x) = x_1^4 + x_2^4 > 0 \quad \forall x \neq 0. \ (= 0 \text{ at } x = 0)$$

$$\dot{V}(x) = 4x_1^3 \dot{x}_1 + 4x_2^3 \dot{x}_2 = 4x_1^3(-x_2^3) + 4x_2^3(x_1^3) = 0$$

Solution curves lie on: $x_1^4 + x_2^4 = c^2$. (enclosing 0)

So, $x_0 = 0$ is a stable, but not asymptotic stable critical point. Note: $Df(0) = 0$. 2 - zero eigenvalues.

Example

$$\dot{x}_1 = -2x_2 + x_2 x_3$$
$$\dot{x}_2 = x_1 - x_1 x_3$$
$$\dot{x}_3 = x_1 x_2$$

0: equilibrium point

$$Df(0) = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_1 = 0, \ \lambda_2 = 2i, \ \lambda_3 = -2i.$$

0: nonhyperbolic.

Try to use ~~the~~ Liapunov Method. Try

$$V(x) = c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2$$

$$\frac{1}{2}\dot{V}(x) = (c_1 - c_2 + c_3) x_1 x_2 x_3 + (-2c_1 + c_2) x_1 x_2$$

Let $c_2 = 2c_1$, $c_3 = c_1 > 0$. $V(x) > 0$. $(x \neq 0)$. $V(0) = 0$

$\dot{V}(x) = 0 \ \forall x$. So, $x = 0$ is stable.

In factor. $x_1^2 + 2x_2^2 + x_3^2 = c^2$.

Example Asymptotically stable, but not a sink.

equilibrium: $x_0 = 0$,

$$\dot{x}_1 = -2x_2 + x_2 x_3 - x_1^3$$
$$\dot{x}_2 = x_1 - x_1 x_3 - x_2^3$$
$$\dot{x}_3 = x_1 x_2 - x_3^3$$

~~The~~ A Liapunov funtion $V(x) = x_1^2 + 2x_2^2 + x_3^2$.

$\dot{V}(x) < 0$. $(x \neq 0)$.

So, asymptotically stable, but $\lambda_1 = 0$, $\lambda_{2,3} = \pm 2i$. not a sink.

For planar nonlinear systems, it is sometimes good to use the polar coordinates, Especially if ~~$Df(x_0)$ has~~ x_0 is a ~~h~~yperbolic equilibrium point, i.e., $Df(x_0)$ has eigenvalues with zero real part.

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases}$$

Assume $P(0,0) = 0$, $Q(0,0) = 0$, i.e., $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium point.

Consider $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$. $\begin{cases} r^2 = x^2 + y^2 \\ \theta = \arctan\frac{y}{x} \end{cases}$.

$$\begin{cases} r\dot{r} = x\dot{x} + y\dot{y} \\ r^2\dot{\theta} = x\dot{y} - y\dot{x} \end{cases}$$

So, the system is
$$\dot{r} = P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta$$
$$r\dot{\theta} = Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta$$

or
$$\frac{dr}{d\theta} = F(r,\theta) \equiv \frac{r[P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta]}{Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta}$$

Example $\begin{cases} \dot x = -y - xy \\ \dot y = x + x^2 \end{cases}$

$$r\dot r = x\dot x + y\dot y = -xy - x^2y + xy + x^2y = 0$$

$$\underline{\dot r = 0 \quad (r>0)}$$

$$r^2\dot\theta = x\dot y - y\dot x = x^2 + x^3 + y^2 + xy^2 = r^2(1+x)$$

$$\dot\theta = 1 + x \quad (r>0) \qquad \underline{\theta' = 1 + r\cos\theta} \quad (r>0)$$

Example $\begin{cases} \dot x = \frac12 x - y - \frac12 (x^3 + y^2 x) \\ \dot y = x + \frac12 y - \frac12 (y^3 + x^2 y) \end{cases}$

$$\begin{cases} \dot r = \frac12 r(1-r^2) \\ \dot\theta = 1 \end{cases}$$

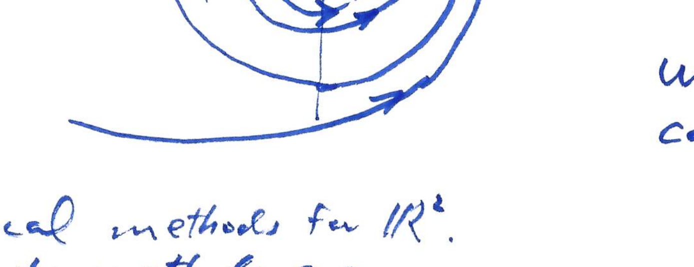
$\dot\theta = 1 \Longrightarrow$ all nonzero solutions spiral around the origin counterclockwise.

$\dot r = \frac12 r(1-r^2) \quad (r>0)$

$r = 1 \Longrightarrow \dot r = 0$. So, $r=1$ is a solution orbit

$0<r<1 \Longrightarrow \dot r > 0 \Longrightarrow$ nonzero solutions spiral away from 0 and toward to the unit circle

$r>1 \Longrightarrow \dot r < 0 \Longrightarrow$ spiral into $r=1$.



$$\dot r = \tfrac12 (r - r^3)$$
$$\dot\theta = 1$$

unit circle is called a *limit cycle*.

Skips topological methods for $\mathbb{R}^2$.
[Not sure if the methods are "mature" for $\mathbb{R}^n$?]

Section 3. Gradient Systems and Hamiltonian Systems

Gradient System

$$\dot{x} = -\nabla U(x), \qquad U: E \to \mathbb{R}^1 \text{ given. } U \in C^1(E).$$

$E \subseteq \mathbb{R}^n$: open.

$$\nabla U(x) = \begin{bmatrix} \frac{\partial U}{\partial x_1} \\ \vdots \\ \frac{\partial U}{\partial x_n} \end{bmatrix} \qquad (x = (x_1, \cdots, x_n) \in E)$$

call $U = U(x)$ a potential of the system (cf. Example on p. 44.)

① Note that for any constant U_0,

$$\dot{x} = -\nabla U(x) \iff \dot{x} = -\nabla\left(U(x) + U_0\right)$$

So, potentials are not unique. One can add a constant not to change the system (sol'ns of the system). If U is bounded below, by adding a constant, we can assume

$$\inf_{x \in E} U = 0.$$

② x_0 is an equilibrium point for $\dot{x} = -\nabla U(x)$
$\iff \nabla U(x_0) = 0$ i.e., x_0 is a critical point of U.

③ If x_0 is an equilibrium point of $\dot{x} = -\nabla U(x)$; $\nabla U(x_0)$, then the linearized system is $\dot{h} = Ah$.

$$A = D(-\nabla U)(x_0) = -\left[\frac{\partial^2 U(x_0)}{\partial x_i \partial x_j}\right]_{i,j=1}^{n} = -\nabla^2 U(x_0)$$

$\nabla^2 U(x_0)$: This is the Hessian matrix of U at x_0. It is always symmetric. So, eigenvalues are always real. and it is always diagonalizable.

④ trajectories $x = x(t)$ $\perp$ level set $U = \text{const}$.
Pf. $\dot{x} = -\nabla U(x) \perp \{U = \text{const}\}$. Q.E.D.

Example $\quad m\ddot{x} + \gamma\dot{x} + \nabla U(x) = 0$
If $m \ll \gamma$ then $\quad \gamma\dot{x} + \nabla U(x) = 0$

$$\dot{x} = -\frac{1}{\gamma}\nabla U(x) = -\nabla\left(\frac{1}{\gamma}U(x)\right).$$

A gradient system.

⑤ $\quad \dot{x} = -\nabla U(x)$

$$\dot{x}\cdot\dot{x} = |\dot{x}|^2 = -\nabla U(x)\cdot\dot{x} = -\frac{d}{dt}\left[U(x(t))\right]$$

$$\int_0^t |\dot{x}(s)|^2 ds = U(x(0)) - U(x(t)) \geq 0$$

$$\Longrightarrow \boxed{U(x(t)) \leq U(x(0)) \quad \forall t \geq 0}$$

Thm ④+⑤. For a gradient system, trajectory $x = x(t)$ moves from high to lower potential, orthogonal to constant potential surfaces.

⑥ Theorem If x_0 is a strictly local minimum of U then it is asymptotically stable.

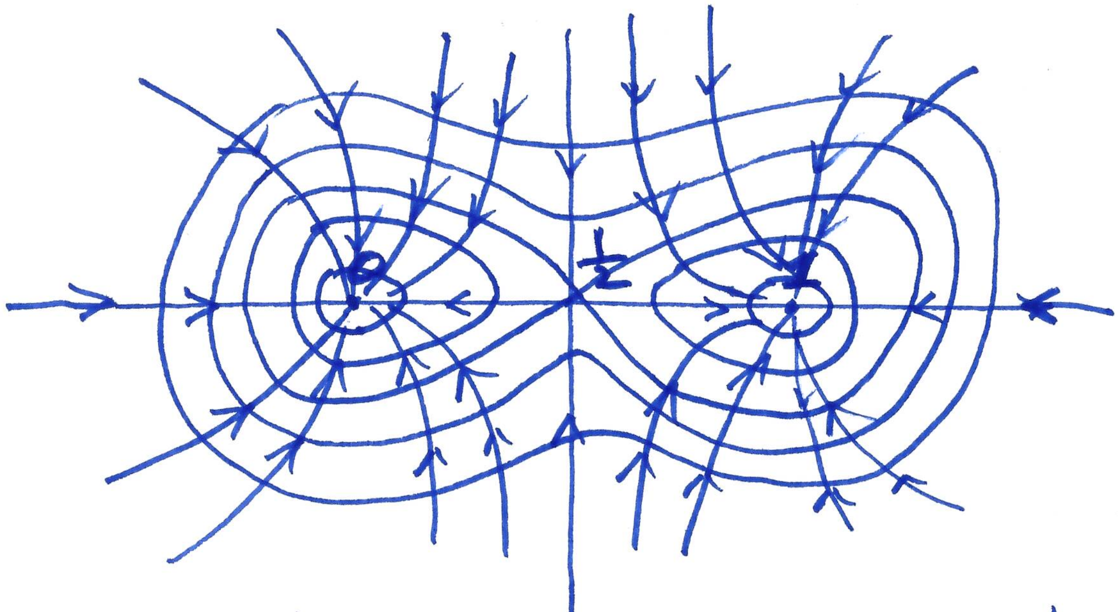
Pf. $U(x) - U(x_0)$ is a (local) $\overbrace{\text{Liapunov function}}^{\text{strict}}$. Q.E.D.

[Local is enough
as the stability is a local property.]

⑦ A corollary: A ~~pote~~ gradient system does not permit a periodic solution (no limit cycles).

Example $\quad U(x,y) = x^2(x-1)^2 + y^2$.

$$\begin{cases} \dot{x} = -4x(x-1)\left(x - \frac{1}{2}\right) \\ \dot{y} = -2y \end{cases}$$



level curves of $U(x,y) = x^4(x-1)^4 + y^2$
and trajectories of $\begin{cases} \dot{x} = -\partial_x U(x,y) \\ \dot{y} = -\partial_y U(x,y). \end{cases}$

critical points : $(0,0)$, $(1/2,0)$, $(1,0)$.
$(0,0)$, $(1,0)$: strict local minima $\Rightarrow$ asymptotically stable.
$(1/2,0)$: saddle point.

$f = -\nabla U$. $\quad Df(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = Df(1,0)$

$Df(\frac{1}{2},0) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

Hamiltonia System

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$$

$x = x(t)$, $y = y(t)$

$H \in C^2(E)$. $E \subseteq \mathbb{R}^{2n}$: open

$H = H(x, y)$

$$\boxed{\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} \\[2mm] \dot{y} = -\dfrac{\partial H}{\partial x} \end{cases}}$$ a Hamiltonia system

$$\frac{\partial H}{\partial y} = \begin{bmatrix} \frac{\partial H}{\partial y_1} \\ \vdots \\ \frac{\partial H}{\partial y_n} \end{bmatrix} \qquad \frac{\partial H}{\partial x} = \ldots \text{ similar.}$$

In stat mech.

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p} \qquad H = H(p, q)$$

Example Consider n particles $x_j = x_j(t)$ $(j = 1, \ldots, n)$ moving around in $\mathbb{R}^3$.

Newton's 2nd law:

$$m_j \ddot{x}_j = -\nabla U(x_1, \ldots, x_n)$$

positions: $q_j = x_j$

momenta: $p_j = m_j \dot{x}_j$

Total energy:

$$\underbrace{\sum_{j=1}^{n} \frac{1}{2} m_j \dot{x}_j^2}_{\text{kinetic}} + \underbrace{U(x_1, \dots, x_n)}_{\text{potential}}$$

Define $H = H(p, q) = H(p_1, \dots, p_n, q_1, \dots, q_n)$

$$= \sum_{j=1}^{n} \frac{1}{2m_j} p_j^2 + U(q_1, \dots, q_n)$$

Then
$$\dot{p} = -\frac{\partial H}{\partial q}$$
$$\dot{q} = \frac{\partial H}{\partial p}$$

<u>Check</u>
$$\dot{p}_j = m_j \ddot{x}_j = -\nabla_{x_j} U(x_1, \dots, x_n) = -\frac{\partial H}{\partial q_j}$$
$$\dot{q}_j = \dot{x}_j = \frac{p_j}{m_j} = \frac{\partial H}{\partial p_j}.$$

<u>Theorem</u> Conservation of energy for $\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases}$.

<u>Pf</u> $\frac{d}{dt} H(x(t), y(t)) = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y}$

$= -\dot{y}\dot{x} + \dot{x}\dot{y} = 0.$ <u>Q.E.D.</u>

We can write $\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases}$ as $\dot{z} = f(z)$

$z = \begin{bmatrix} x \\ y \end{bmatrix}$ $f(z) = \begin{bmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{bmatrix}$. Then $\boxed{\nabla \cdot f(z) = 0}$

<u>Liouville's thm</u> conservation of volume under $\phi_t(x)$, the flow of ~~the~~ an Hamiltonian system.

Example $m\ddot{x} = f(x)$ $\quad x = x(t) \in \mathbb{R}^1$. $f \in C^1(\mathbb{R})$

Take $m = 1$.

$\ddot{x} = f(x)$ $\quad$ [Not $\dot{x} = f(x)$]

Let $U(x) = -\int_0^x f(s)\,ds$. $\quad -U'(x) = f(x)$.

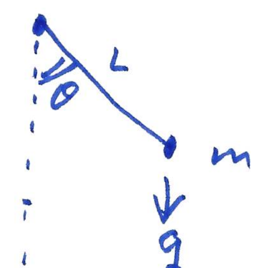
$\ddot{x} = -U'(x)$ $\quad$ (Not a gradient system)

Reformulate: $\begin{cases} \dot{x} = y \\ \dot{y} = f(x). \end{cases}$

Let $H(x,y) = T(y) + U(x) = \frac{1}{2}y^2 + U(x)$

Then $\begin{cases} \dot{x} = \partial_y H \\ \dot{y} = -\partial_x H. \end{cases}$

Example Pendulum



$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Let $\tau = \omega t = \sqrt{\frac{g}{L}}\,t$

change t to τ, still use θ.

$$\ddot{\theta} + \sin\theta = 0$$

Let $v = \dot{\theta}$ (dimensionless) angular velocity

$$\begin{cases} \dot{\theta} = v \\ \dot{v} = -\sin\theta \end{cases}$$

$$H(v,\theta) = \frac{1}{2}v^2 - \cos\theta = \frac{1}{2}v^2 + U(\theta)$$

$$\begin{cases} \dot{\theta} = \frac{\partial H}{\partial v} \\ \dot{v} = -\frac{\partial H}{\partial \theta} \end{cases}$$ a Hamiltonian system.

$\frac{d}{d\tau} H(v(\tau), \theta(\tau)) = 0 \Rightarrow H(v,\theta) = \text{const}$.

Critical points for $\begin{cases} \dot{\theta} = v \\ \dot{v} = -\sin\theta \end{cases}$

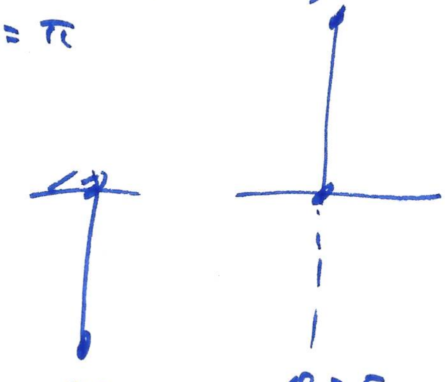
$(v, -\sin\theta) = (0, 0)$. $v = 0$. $\theta = k\pi$ $(k \in \mathbb{Z})$

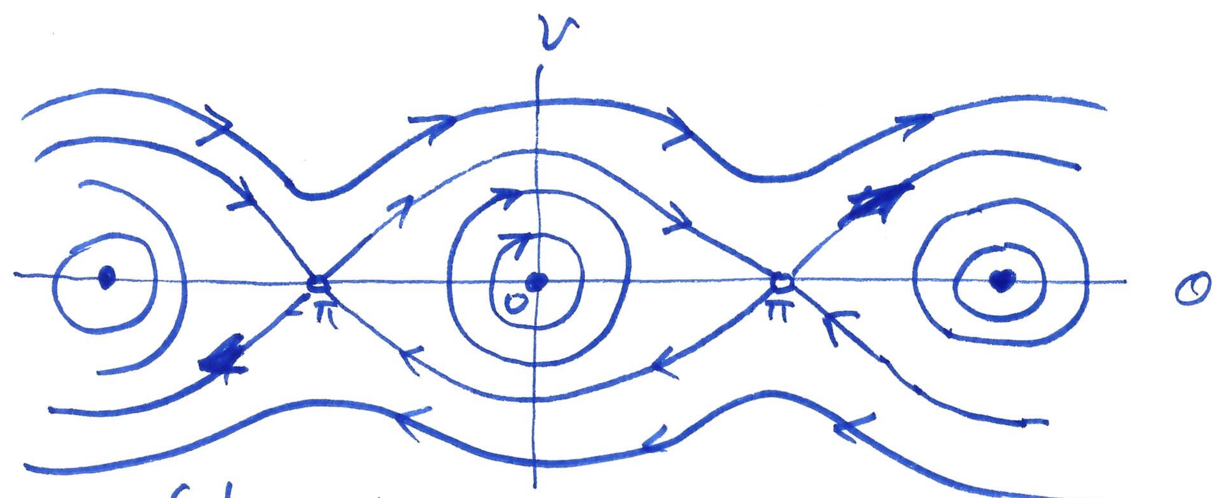
Only consider $\theta = 0$, or $\theta = \pi$

$A(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\lambda^2 + 1 = 0$, $\lambda_{1,2} = \pm i$

$(0,0)$: linear center

$A(\pi, 0) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\lambda_1 = -1$. $\lambda_2 = 1$.





When a (planar) ~~gradient~~ system ~~is a~~ $\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases}$ is Hamiltonian?

<u>Thm</u> ① $\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases}$ is Hamiltonian $\Longleftrightarrow$ ② $\begin{cases} \dot{x} = Q(x, y) \\ \dot{y} = -P(x, y) \end{cases}$ is a gradient system.

<u>Pf</u>. $\begin{cases} \dot{x} = P(x, y) = \dfrac{\partial H}{\partial y} \\ \dot{y} = Q(x, y) = -\dfrac{\partial H}{\partial x} \end{cases} \Longleftrightarrow \begin{matrix} -Q = \dfrac{\partial H}{\partial x} \\ P = \dfrac{\partial H}{\partial y} \end{matrix}$

$\Longleftrightarrow \begin{cases} \dot{x} = Q(x, y) \\ \dot{y} = -P(x, y) \end{cases} = -\nabla H(x, y)$ <u>Q.E.D.</u>

Note: ①. ② : same critical pts. orthogonal trajectories. etc.