Part 3. Global Theory

Section 3.1 Concepts: Dynamical Systems, Global Existence, Limit Cycles, and Attractors

Section 3.2 Periodic Structure

Section 3.3 Planar Systems

Section 3.1 Same basic concepts

Consider $\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$

Local existence: $x = x(t)$ $x(0, B)$ $\subset C^1(0, 3)$ $\subset C^1$ (max. interval of solution)

Remember: $\begin{cases} x' = x^2 \\ x(0) = 1 \end{cases}$

It will be convenient if we can find some equivalent system for which the solution exists for all $t \in (-\infty, 0)$.

Definition Let $E \neq \phi$ be an open subset of $\mathbb{R}^n$, $f \in C^1(E, \mathbb{R}^n)$. A $\phi \in C^1(\mathbb{R} \times E, E)$ is dynamical system, if $\phi_t(\phi_s(x))$ satisfies

(i) $\phi_0(x) = x$ $\forall x \in E$, and

(ii) $\phi_t \circ \phi_s(x) = \phi_{t+s}(x)$ $\forall s, t \in \mathbb{R}$ $\forall x \in E$. 
Remarks

1. \( \phi_t \circ \phi_t = \text{Id} \).
   So, \( \{ \phi_t \}_{t \in \mathbb{R}} \) is a one-parameter family of
diffeomorphisms, forming a commutative
group under composition.

2. \( \forall x \in E \). Define
   \[
   f(x) = \frac{\partial}{\partial t} \phi_t(x) \big|_{t=0}.
   \]
   Then \( \phi_t(x_0) \) solves \( \begin{cases} x' = f(x), \\ x(0) = x_0 \end{cases} \).

   Unique solution, with the max. interval of
   existence \( I(x_0) = (-\infty, \infty) \).

3. Example: Let \( A \) be a real matrix, be given.
   \[
   \begin{cases}
   x' = Ax & \Leftrightarrow x(t) = x_0 e^{At} \\
   x(0) = x_0
   \end{cases}
   \]
   So, \( \phi(t, x) = x e^{At} \) is a dynamical system
   on \( E = \mathbb{R}^n \).

Definition: Let \( E_1, E_2 \) be open in \( \mathbb{R}^n \). \( f \in C^1(E_1, \mathbb{R}^m) \)
and \( g \in C^1(E_2, \mathbb{R}^m) \). The two systems

\( (1) \ x' = f(x) \) \quad and \quad \( (2) \ x' = g(x) \)

are **topologically equivalent** if \( \exists \) homeomorphism

\( H : E_1 \rightarrow E_2 \) (one-to-one, onto, continuous)

that maps trajectories of \( (1) \) to Those of \( (2) \)
and preserves their orientation by time.

More precisely, let \( \phi_t(x) \), \( \psi_t(x) \) be associated
with \( (1) \) and \( (2) \), respectively. Then, \( \forall x \in E_1 \),
There exists a continuously differentiable function $f: \mathbb{R} \to \mathbb{R}$ such that
\[
\frac{dx}{dt} > 0 \quad \forall \tau. \quad \text{(orientation preserving)}
\]
and
\[
H_0 \phi_{\tau}(x, t) = \frac{1}{\tau} \circ H(x) \quad \forall \tau \in \mathbb{R}, \quad \forall x \in \mathbb{R}.
\]
First talk about Thm on 1.5.3. Then this Thm.

**Theorem (Global existence by rescaling time)**

Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. For $x_0 \in \mathbb{R}^n$,

\[
\begin{align*}
& \dot{x} = \frac{f(x)}{1 + |f(x)|} \\
& x(0) = x_0
\end{align*}
\]

has a unique solution $x(t)$ defined for all $t \in \mathbb{R}$, i.e., it defines a dynamical system, and this system is topologically equivalent to $\dot{x} = f(x)$.

How the time is rescaled?

Set:
\[
T = \int_0^t \left[ 1 + |f(x(s))| \right] ds.
\]

Now
\[
\frac{dx}{d\tau} = \frac{dx}{dt} / \frac{dt}{d\tau} = \frac{f(x)}{1 + |f(x)|} \quad \text{if} \quad \frac{dx}{dt} = f(x).
\]

**Proof of Theorem (Sketch)**

\[
x(t) = x_0 + \int_0^t \frac{f(x(s))}{1 + |f(x(s))|} ds
\]

\[
\Rightarrow |x(t)| \leq |x_0| + \int_0^t |f(x(s))| ds = |x_0| + t |f|_{\max} \quad \forall + c(x, \beta), \ max.
\]

\[
\text{interval for solution to } x(t).
\]

Show $\beta = \infty$ (then some $x = -\infty$). If $\beta < \infty$

then $|x(t)| \leq |x_0| + \beta \quad \forall + c(x, \beta, \beta)$. So

Previous solution $x(t), (t \in [0, \beta]) \in \mathcal{K} = \{ x: |x| \leq \beta \}$

\[
\Rightarrow \beta = \infty. \quad \text{Q.E.D.}
\]
Example \( \begin{cases} \dot{x} = x^2 \\ x(0) = x_0 \end{cases} \) \( x(t) = \frac{x_0}{1 - x_0 t} \)

\[ \begin{cases} x = \frac{x^2}{1 + x^2} \\ x(0) = x_0 \end{cases} \]

\[ x(t) = A + \frac{x_0}{1 + x_0 t} \pm \frac{x_0}{1 + x_0 t} \sqrt{1 + \frac{2x_0^2}{(1 + x_0 t)^2}} \]

\[ \forall t \in (-\infty, \infty) \]

\[ \exists 0 \text{ if } x_0 = 0 \]

\[ T(t) = A + \frac{x_0}{1 - x_0 t} \]

If \( f \in C'(E, K^m) \) or \( C'(K^m, M^m) \) then the rescaling \( \frac{f(x)}{1 + |f(x)|} \) may not work. Check

\[ \begin{cases} \dot{x} = \frac{1}{2 x} \\ x(0) = x_0 \end{cases} \]

the rescaled system is \( \begin{cases} \dot{x} = \frac{1}{2 x^2} \\ x(0) = x_0 \end{cases} \)

and \( T(x) = \frac{-1}{(x_0 + \frac{1}{2})^2} x_0 \) not \( R \).

But, we have

Theorem \( \forall f \in C'(E, K^m) \exists F \in C'(E, K^m) \) i.e.

1. \( \dot{x} = F(x) \) defines a dynamical system (i.e., solution \( x = x(t) \) is defined on \( t \in \mathbb{R} \));

2. \( \dot{x} = \overline{F(x)} \) and \( \dot{x} = f(x) \) are topologically equivalent.

Skip the proof.
Theorem. If $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the global Lipschitz condition: \[ |f(x_1) - f(x_2)| \leq K |x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{R}^n, \]
then $\forall x_0 \in \mathbb{R}^n$ the system $\{ \dot{x} = f(x) \}$ has a unique solution defined on $t \in (-\infty, \infty)$.

Proof. Similar idea. First: $\frac{d}{dt}|x(t)| \leq |x'(t)| = |f(x(t))|$

So, $\frac{d}{dt}|x(t) - x_0| \leq |x'(t)| = |f(x(t))|$

$\leq |f(x(t)) - f(x_0)| + |f(x_0)|$

$\leq K |x(t) - x_0| + |f(x_0)|$

If the max. soln. interval is $I(x_0) = (a, \beta)$ with $\beta < \infty$, then with $g(t) = |x(t) - x_0|$, we have

$g(t) = \int_a^t \frac{dg(s)}{ds} ds \leq \int_a^t |f(x_0)| + K \int_0^s |g(s)| ds$

$\Rightarrow |x(t) - x_0| \leq |f(x_0)| + K \int_0^t |g(s)| ds$

$\forall t \in [0, \beta)$. Gronwall's inequality.

$\Rightarrow |x(t) - x_0| \leq e^{Kt} |f(x_0)| + K \int_0^t e^{K(s-t)} |g(s)| ds$

$\forall t \in [0, \beta)$. Again $\Rightarrow \beta = \infty$.  \(\Box.15.1\).

Theorem (Chillingworth). Let $M$ be an $n$-dimensional compact manifold. Then $\dot{x} = f(x)$, $\quad \dot{x} = f(x)$

has a unique solution on $t \in (-\infty, \infty)$.  \(\Box.15.11\).
Limit Sets and Attractors

Let $E \subseteq \mathbb{R}^n$ be open. Assume $f \in C'(E, \mathbb{R}^n)$ and
\[
\begin{cases}
\dot{x} = f(x) \\
x(0) = x_0
\end{cases}
\]
has a unique solution $x = x(t) = \phi_t(x_0)$ defined on $t \in (-\infty, \infty)$, i.e., assume $\dot{x} = f(x)$ defines a dynamical system $\phi_t(x) \ (x \in E, t \in \mathbb{R})$.

Notation: $\forall x_0 \in E$
\[
P_{x_0} = \{ x \in E : x = \phi(t, x_0), t \in \mathbb{R} \} = \{ \phi(t, x_0) : t \in \mathbb{R} \}.
\]
- solution curve or trajectory

Also, $P_{x_0}^+ = \{ \phi(t, x_0) : t \geq 0 \}$
\[
P_{x_0}^- = \{ \phi(t, x_0) : t \leq 0 \}.
\]
$\Gamma = P_{x_0}^+ \cup P_{x_0}^-$. (Point) of $\phi_t(x_0) \cap \{ \phi(t, x) \}$

Definition: $p \in E$ is an $\omega$-limit $x$ if $\exists t_n \to \infty$ s.t.
\[
\lim_{t \to \infty} \phi(t_n, x) = p.
\]
$\in E$ is an $\alpha$-limit of $\phi(t, x)$ if $\exists t_n \to \infty$ s.t.
\[
\lim_{t \to \infty} \phi(t_n, x) = q.
\]
$\omega(\Gamma) = \omega$-limit set of $\Gamma$
\[
= \{ \omega$-limit points of $\phi(t, x), x \in \Gamma \}
\]
$\alpha(\Gamma) = \alpha$-limit set of $\Gamma$
\[
= \{ \alpha$-limit points of $\phi(t, x), x \in \Gamma \}
\]
$\omega(\Gamma) \cup \alpha(\Gamma):$ limit set of $\Gamma$
Theorem 1 \(\alpha(p), \omega(p)\) are closed in \(E\).
2 If \(p\) is bounded then \(\alpha(p), \omega(p)\) are non-empty, connected, and compact subsets of \(E\).

\[ \text{Theorem} \quad \forall p \in \omega(p) \Rightarrow \Gamma_p \subseteq \omega(p), \]
\[ \forall p \in \alpha(p) \Rightarrow \Gamma_p \subseteq \alpha(p). \]

In particular, \(\alpha(p), \omega(p)\) are invariant w.r.t. the flow \(\phi_t\) defined by \(\dot{x} = f(x)\): \(\phi_t(\alpha(p)) \subseteq \alpha(p)\), \(\phi_t(\omega(p)) \subseteq \omega(p)\).

This one is intuitively true; no skip the proof.

A \(\subseteq E\) is invariant if \(\phi(A) \subseteq A\).

Attracting set of \(\dot{x} = f(x)\): closed invariant set \(A \subseteq E\) s.t. \(\exists\) a neighborhood of \(A\), called \(U\), s.t.
\[ x \in U \Rightarrow \phi(t) \subseteq U \quad \forall t \geq 0 \text{ and } x(t) \rightarrow A \text{ as } t \rightarrow \infty. \]

An attractor: an attracting set containing a dense orbit (trajectory).

Example \(\{\dot{r} = r(1-r^2), \quad \dot{\theta} = 1\}\)

Let \(P_0\): counterclockwise flow on unit circle
\(P_0\): an attractor, \(\alpha(P_0) = P_0, \omega(P_0) = P_0\).

In fact, \(P_0\) is a stable limit cycle.
Example \[
\begin{align*}
x' &= -y + x(1 - x^2 - y^2) \\
y' &= x + y(1 - x^2 - y^2) \\
z' &= 0
\end{align*}
\]
\(z = 0\): planar motion.
So, \(z = \text{const.}\) invariant.
\(S^2 \cup \{(0,0,t) : |t| \geq 1\}\) is an attracting set.

Example \[
\begin{align*}
x' &= -y + x(1 - x^2 - y^2) \\
y' &= x + y(1 - x^2 - y^2) \\
z' &= \alpha \quad (\alpha = \text{const.} \neq 0)
\end{align*}
\]
The \(z\)-axis \(\cup\) cylinder \(x^2 + y^2 = 1\) is invariant, the cylinder is an attracting set.

Example An invariant torus as an attracting set.

Example The Lorenz system \[
\begin{align*}
x' &= \sigma(y - x) \\
y' &= \rho x - y - xz \\
z' &= -\beta z + xy
\end{align*}
\]
Google search this system!

Example \(1-d\) \[
\begin{align*}
x' &= -x^4 \sin(\frac{\pi}{x}) \quad \text{critical pt.} \quad o_+ = \frac{1}{h}, \quad h \in \mathbb{Z}^+ \\
0^{-1} &:\text{ an attracting set.} \quad \pm \frac{1}{2h} (n=1,2,\ldots) \quad \text{repelling.} \\
0^{+1} &:\text{ an attracting set.} \quad \pm \frac{1}{2h-1} (n=2,3,\ldots) \quad \text{attracting.}
\end{align*}
\]
Section 2: Periodic Structures, Periodic Orbits, Limit Cycles, and Separatrix Cycles

Notation: \( E \subset \mathbb{R}^n \) open, \( f \in C^1(E, \mathbb{R}^n) \). \( \phi(t, x) \) is the dynamic system defined by \( \dot{x} = f(x) \). \( \ldots \) \( (1) \)

Definition: A closed orbit or cycle of (1) is a closed solution curve of (1) that is not an equilibrium point of (1).

For a cycle \( \Gamma \), for any \( \varepsilon > 0 \), \( U_\varepsilon \) stable: If \( \exists \) a neighborhood \( \mathcal{V} \) of \( \Gamma \) s.t.
\[ x \in U_\varepsilon \Rightarrow \text{dist}(\phi(t, x), \Gamma) < \varepsilon \quad \forall t \geq 0 \]
unstable: If not stable.

Asymptotically Stable

If it is stable and
\[ \exists \text{n.h.b. } U \text{ of } \Gamma \text{ s.t. } x \in U \Rightarrow \lim_{t \to \infty} \text{dist}(\phi(t, x), \Gamma) = 0. \]

Proposition: Any asymptotically stable cycle is an attractor.

The Poincaré Map (or first return map)

(A tool for studying the stability and bifurcations of periodic orbits.)

\[ x \rightarrow \phi(x) \]

\[ \Sigma \rightarrow \Sigma \]

\( P: \) closed orbit
\( x_0 \in \Gamma \) hyperplane
\[ \Sigma \ni \Gamma \text{ at } x_0 \]
\( x \) near \( x_0 \)
\( x \in \Sigma \)
\( \Rightarrow 1 \text{st return } \phi(x) \notin \Sigma. \]
Theorem  Let \( x_0 \in E \). Suppose \( \Phi(t, x) \) is \( T \)-periodic and \[
P = \{ x \in \mathbb{R}^n : x = \Phi(t, x_0), \ 0 \leq t \leq T \} \subseteq E, \]
Let \[
\Sigma = \{ x \in \mathbb{R}^n : (x - x_0), f(x_0) = 0 \}. \]
[In \( \mathbb{R}^n \) that is perpendicular to \( P \) at \( x_0 \).]
Then \( \exists \delta > 0 \) and a unique \( \tau \in C^1(\mathbb{R} \times [0, \delta]) \) such that \( \tau(0, x_0) = T \) and \( \Phi(\tau(x, x_0), x_0) \in \Sigma \ \forall x \in B(x_0, \delta) \).
The map \( \tau(x) = \Phi(\tau(x, x_0), x_0), \ \forall x \in B(x_0, \delta) \cap \Sigma \), is called the Poincaré map for \( P \) at \( x_0 \).

Proof Use the Implicit Function Theorem (IFTM).
Define \( F(t, x) = [\Phi(t, x) - x_0], f(x_0) \). \( F \in C^1(\mathbb{R} \times E) \), and \( F(0, x_0) = 0 \).
Since \( \Phi(t, x_0) = \Phi(t, x_0) \) solves \( \{ x^* = f(x), (x, x_0) = x_0 \} \), we have \( \frac{\partial F(t, x_0)}{\partial t} = \frac{\partial \Phi(t, x_0)}{\partial t} - f(x_0) = 0 \), \( f(x_0) = f(x_0) f(x_0) = f(x_0) \), \( x_0 \) is not an equilibrium point.

Now, IFTM \( \exists \delta > 0 \). \( \exists \tau \in C^1(\mathbb{R} \times [0, \delta]) \) such that \( \tau(x) = T \) and \( F(\tau(x, x_0), x_0) = 0 \ \forall x \in B(x_0, \delta) \). Hence, for any \( x \in B(x_0, \delta) \), \( \tau(\Phi(t, x), x_0) \in \Sigma \). \( \tau(0, x_0) = T \), i.e., \( \Phi(\tau(x, x_0), x_0) \in \Sigma \).

We now present one example of computing the Poincaré maps. We then describe the idea of using the Poincaré maps to study the stability of a cycle in \( \mathbb{R}^3 \). After that, we state the general stability results for \( \mathbb{R}^n \).

First, let's introduce the concept of limit cycles (in \( \mathbb{R}^3 \))
A limit cycle is an isolated closed trajectory. Here, isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle.

![Stable Limit Cycle](Image)

**Stable Limit Cycle**

**Unstable Limit Cycle**

**Half-Stable Limit Cycle**

Same as asymptotically stable limit cycle. Describe self-sustained oscillations.

Examples of stable limit cycles: heart beating, periodic firing of a pacemaker neuron, daily rhythms of human body temperatures and hormone secretion, and dangerous self-excited vibrations in bridges and airplane wings.

Proposition

- Linear systems have no limit cycles.

\[ \dot{x} = Ax \]

- Periodic solutions are not isolated.

- If \[ x = x(t) \] is periodic,

\[ \Rightarrow \dot{x} = c x(t) \]

is also periodic.

- So, limit cycles are nonlinear phenomena.

- Gradient systems do not permit limit cycles.

\[ \dot{x} = -\nabla U(x) \]
Example \[
\begin{align*}
\dot{x} &= -y + x(1 - x^2 - y^2) \\
\dot{y} &= x + y(1 - x^2 - y^2)
\end{align*}
\] \[\iff \quad \dot{r} = r(1-r^2) \] \[\dot{\theta} = 1\]

Limit cycle at \(r=1\)

Construct the Poincaré map for the limit cycle \(r=1\).

at \((r_0, \theta_0) = (r_0, \theta_0)\).

First, solve \(\dot{r} = r(1-r^2)\) to get
\[r(t, r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-t} \right]^{-\frac{1}{2}}\]

so \(\dot{\theta} = 1\) to get
\[\theta(t, \theta_0) = t + \theta_0\]

So, for \(\Sigma : \theta = \theta_0\) s.t. \((r_0, \theta_0) \in \Sigma \cap \Gamma\) at \(t = 0\),

intersects \(\theta = \theta_0\) again at \(t = 2\pi\). Thus,

\[P(r_0) = \left[1 + \frac{1}{r_0^2} \right]^{-\frac{1}{2}} = \frac{1}{\sqrt{1 + \frac{1}{r_0^2}}} \] \(\text{The Poincaré map.}\)

In particular, \(P(1) = 1\).

Note that
\[P'(r_0) = e^{-a\pi} r_0^{-3} \left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-a\pi}\right]^{-3/2}\]
\[P'(1) = e^{-a\pi} < 1\]
This in fact implies the stability.
Consider $\mathbb{R}^2$. Let $x_0 = 0 \in \Gamma \cap \Sigma$. \( \rho \): cycle. \( \Sigma = \Sigma^- \cup \Sigma^+ \)

\[ s > 0 \implies \Sigma^+ \]
\[ s < 0 \implies \Sigma^- \]
\[ \rho(0) = 0. \]

\( \rho \): the Poincaré map of \( \Gamma \) at \( 0 \), defined on \(-\delta, \delta\]

Introduce the displacement function

\[ d(s) = \rho(s) - s. \]

Then \( d(0) = \rho(0) - 1 \neq 0 \).

The Mean-Value Thm \( \implies \exists \theta \) in between \( 0 \leq s \leq 1 \).

\[ d(s) = d'(0) s + o(s) \text{ \ if \ } |s| < 1 \]

\[ d(s) = d'(0) s \]
\[ d'(0) s + o(s) = d'(0) s + o(s) \text{ \ as \ } |s| < 1. \]

Thus

\[ d'(0) < 0 \implies d(s) < 0 \text{ \ for \ } s > 0 \implies \rho \text{ \ is \ stable} \]
\[ d'(0) > 0 \implies d(s) < 0 \text{ \ for \ } s < 0 \implies \rho \text{ \ is \ unstable} \]

A beautiful idea!

A remark: if \( d(0) = d'(0) = \ldots = d^{(k-1)}(0) = 0 \), \( d^{(k)}(0) \neq 0 \).

Then \( d^{(k)}(0) < 0 \implies \text{stable} \]
\( d^{(k)}(0) > 0 \implies \text{unstable} \)

If \( k \) is odd, semi-stable if \( k \) is even.
Theorem. Let \( \gamma(t) \) be a periodic solution of \( \dot{x} = f(x) \) of period \( T \). Let \( P(x) \) be the Poincaré map along a straight line normal to \( \Gamma = \{ x \in \mathbb{R}^n : x = \gamma(t), 0 \leq t \leq T \} \) at \( x_0 \). Then
\[
P'(0) = \exp \int_0^T \nabla f(\gamma(t)) \, dt.
\]
Moreover,
\[
\int_0^T \nabla f(\gamma(t)) \, dt \begin{cases} < 0 \Rightarrow \gamma(t) \text{ is a stable limit cycle.} \\
> 0 \Rightarrow \gamma(t) \text{ is an unstable limit cycle.} \\
= 0 \Rightarrow \gamma(t) \text{ may be a stable, unstable, or semi-stable limit cycle or it may belong to a continuous band of cycles.}
\end{cases}
\]

Example
\[
\begin{align*}
\dot{x} &= x(1-x^2) \\
\dot{y} &= 1
\end{align*}
\]
\[
\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad T = 2\pi
\]
or
\[
\begin{align*}
\dot{x} &= -y + x(1-x^2-y^2) \\
\dot{y} &= x + y(1-x^2-y^2)
\end{align*}
\]
\[
\nabla f(x, y) = 2 - 4x^2 - 4y^2
\]
\[
\int_0^{2\pi} \nabla f(\gamma(t)) \, dt = \int_0^{2\pi} (2 - 4 \cos^2(t) - 4 \sin^2(t)) \, dt = -4\pi < 0
\]
\[
\Rightarrow P'(0) = -2\pi \in (-\pi, 0) \quad \gamma(t) \text{: stable limit cycle.}
\]

For general \( \mathbb{R}^n \), let \( \dot{x} = f(x) \), suppose \( P: \gamma(t) \)

is a periodic orbit, with period \( T \). Let \( P \) be the Poincaré map of \( \gamma(t) \) at \( x_0 \). Then \( DP(x) \) is \( (n-1) \times (n-1) \)
matrix. If \( \| DP(x_0) \| < 1 \) then \( P \) is asymptotically stable.

If \( P \) is given by \( \phi(x) = \phi(t, x) \), \( x_0 = 0 \in \Gamma \), the \( x_n \)-axis is
tangent to \( \Gamma \) at \( 0 \), pointing same direction as the motion along \( \Gamma \), then \( DP(0) = \left[ \frac{\partial \phi}{\partial x_j}(T, 0) \right]_{j=1}^{n-1} \).
The $n-1$ eigenvalues of $D\Phi(0)$ are $e^{\lambda_1 t}, \ldots, e^{\lambda_{n-1} t}$.

$0, \lambda_1, \ldots, \lambda_{n-1}$ are eigenvalues of a constant non-zero matrix $B$:

$$A(t) = D\Phi(x(t)) = D\Phi(x(t)) \cdot x(t) \quad \text{cycle}$$

$\dot{x} = A(t)x$

has the fundamental matrix $E(t)$ defined by

$$\begin{align*}
E(t) &= A(t) E(t) \\
E(0) &= I
\end{align*}$$

**Floquet's Theorem** $\Rightarrow \ E(t) = e^{Bt}$. This defines $B$.

Periodic Eigenvalues of $B$ are $0, \lambda_1, \ldots, \lambda_{n-1}$. These define stable, unstable, and center manifolds of $P$; they are invariant sets of $\Phi(t)$.

A general result: $E \subseteq \mathbb{R}^n$ open. $f \in \mathcal{C}^1(E, \mathbb{R}^n)$. $\dot{x} = f(x)$ defines a dynamical system $\Phi(t, x) = \Phi(t, x)$.

**Theorem** If $x(t)$ is a periodic orbit with period $T$ then $\int_0^T Df(x(t)) dt > 0 \Rightarrow x(t)$ is not asymptotically stable.

**Remark** For $n \geq 3$, $\int_0^T Df(x(t)) dt < 0$ does not imply that $x(t)$ is asymptotically stable.
Sections. Planar Systems: Stability of
Periodic Solutions, applications, etc.

In this section, we assume:
1. \( E \subseteq \mathbb{R}^2 \): open (non-empty)
2. \( f \in C^1(E, \mathbb{R}^2) \)
3. \( \dot{x} = f(x) \) defines a dynamical system
   \( \varphi_t(x) = \varphi(tx) : \mathbb{R} \times E \to E, \varphi \in C^1 \).

Theorem (Poincaré-Bendixson) Let \( \Gamma \) be a
trajectory of \( \dot{x} = f(x) \) with \( \Gamma^+ \) contained in a compact
subset \( K \) of \( E \). Then \( \omega(\Gamma) \) contains no critical
point of \( \dot{x} = f(x) \). Moreover, \( \omega(\Gamma) \) is a periodic
orbit of \( \dot{x} = f(x) \).

A different version of Poincaré-Bendixson Theorem.

Assume
1. \( \exists \) a compact set \( K \subseteq E \) that
does not contain any critical
   points of \( f \).
2. \( \exists \) a trajectory \( \Gamma^+ \) contained in \( K \).
   \( \forall x \in \Gamma^+ \exists \tau \in \mathbb{R} \), \( \varphi(\tau, x) \in K \) \( \forall \tau \geq 0 \).

Then, \( \Gamma \) is a closed orbit or it spirals toward
a closed orbit as \( t \to \infty \).
The trapping region method.

\[ K \text{ compact, connected on the boundary of } K. \]

\[ \Rightarrow \text{all trajectories are confined in } K. \]

If \( \exists \) no critical points in \( K \), then \( K \) contains a closed orbit.

**Example**

\[
\begin{align*}
\dot{r} &= r(1-r^2) + \mu r \cos \omega \\
\dot{\omega} &= 1
\end{align*}
\]

\( \mu = 0 \): stable limit cycle at \( r = 1 \).

We show now that \( 0 < \mu < c \) \( \Rightarrow \) a closed orbit exists.

**Solution.** We find \( r_{\text{min}}, r_{\text{max}}: 0 < r_{\text{min}} < r_{\text{max}} < 1 \).

We shall consider \( K = \{ r: r_{\text{min}} \leq r \leq r_{\text{max}} \} \)

with \( \dot{r} < 0 \) at \( r = r_{\text{max}} \), \( \dot{r} > 0 \) at \( r = r_{\text{min}} \), and \( K \) has no critical points of the system.

For \( r_{\text{min}} \): require \( \dot{r} = r(1-r^2) + \mu r \cos \omega > 0 \)

for all \( \omega \).

\[
(1-r^2) + \mu r \cos \omega \geq r(1-r^2) + \mu r > 0
\]

\( 1-r^2-M > 0 \), \( r_{\text{min}} < \sqrt{1-M} \) \((0 < \mu < 1)\)

Similarly, \( r_{\text{max}} > \sqrt{1+M} \).

\[ \text{e.g. } r_{\text{min}} = 0.999 \sqrt{1-M}, \quad r_{\text{max}} = 1.001 \sqrt{1+M}. \]

In \( \{ r_{\text{min}} < r < r_{\text{max}} \} \), no crpts. So a closed orbit exists.
Example: A model for glycolysis - a biochemical process.  
\[ x = ADP, \quad y = ATP \]  
\[ \begin{align*}  
\dot{x} &= -x + ay + x^2y \\
\dot{y} &= b - ay - x^2y 
\end{align*} \]  \((a, b > 0)\)

The small disk contains a crit. pt.

\[ \kappa = \quad - \quad O \]

Liénard Systems

1. \( \ddot{x} + f(x) \dot{x} + g(x) = 0 \)

Generalization of the van der Pol oscillator

2. \( \dot{x} + \mu(x^2 - 1) \dot{x} + x = 0 \)

Rewrite (1):  
\[ \begin{align*}  
\dot{x} &= y \\
\dot{y} &= -g(x) - f(x) y
\end{align*} \]
Liénard's Theorem

Suppose

1. \( f, g \in C^1 \)
2. \( f \) is even: \( f(-x) = f(x) \)
3. \( g \) is odd: \( g(-x) = -g(x) \), and \( g(x) \to 0 \) as \( x \to 0 \)
4. \( f(x) = \int_0^x f(t) \ dt \) has exactly one positive zero \( a > 0 \), negative in \( 0 < x < a \), positive and non-decreasing for \( x > a \), and \( \lim_{x \to \infty} f(x) = 0 \).

Then (3) has a unique, stable limit cycle surrounding the origin in \( \mathbb{R}^2 \).

\[ F \]

\[ F \text{ is odd.} \]

Example. For the van der Pol equation

\[ f(x) = m(x^2-1), \quad g(x) = x \]
\[ f(x) = m \left( \frac{1}{3} x^3 - x \right) = \frac{1}{3} mx (x^2 - 3), \quad a = \sqrt{3} \]

So, Thm \( \Rightarrow \) 3 unique, stable limit cycle.

We now analyse the van der Pol system for \( M \gg 1 \):

\[ x'' + M x (x^2-1) = \frac{df}{dx} (x + M (\frac{1}{3} x^3 - x)) \]

Let \( f(x) = \frac{1}{3} x^3 - x, \quad w = x' + M (\frac{1}{3} x^3 - x) \)

\[ \Rightarrow w = \dot{x} + M x (x^2-1) = -x \]

So, equivalently

\[ \begin{cases} x'' = w - M f(x) \\ w = -x \end{cases} \]
Change variable: \( y = \frac{w}{u} \).

\[
\begin{align*}
\dot{x} &= \mu \left( y - f(x) \right) \\
\dot{y} &= -\frac{\mu}{x} x
\end{align*}
\]

\( y - f(x) = 0 \implies y = \frac{1}{3} (x^3 - a) \)

\( \mathcal{M} \gg 1 \): fast horizontal motion
slow vertical motion.

Estimate the period of the limit cycle for \( \mathcal{M} \gg 1 \):
The period \( T \approx \) time for travelling along the two slow branches.

\[
\begin{align*}
y &= f(x) \\
\dot{y} &= f'(x) \quad x' = (x^{\frac{1}{3}} - 1) x' \\
\text{But,} \quad \dot{y} &= -\frac{x}{\mathcal{M}} \\
\text{So,} \quad x' &= -\frac{x}{\mathcal{M}(x^{\frac{1}{3}} - 1)}
\end{align*}
\]

\[
\frac{dx}{dt} \approx -\frac{x}{\mathcal{M}(x^{\frac{1}{3}} - 1)} \quad dt \approx -\frac{\mathcal{M}(x^{\frac{1}{3}} - 1)}{x} dx
\]

\( x_A = 2 \), \( x_B = 1 \).

\[
T = 2 \int_{x_A}^{x_B} -\frac{\mathcal{M}}{x} (x^{\frac{1}{3}} - 1) dx = 2 \mathcal{M} \left[ \frac{x^{\frac{4}{3}}}{4} - 2x \ln x \right]_{2}^{1} = \mathcal{M} (3 - 2 \ln 2) = O(\mathcal{M}).
\]
Weekly nonlinear oscillators

\[ \ddot{x} + x + 3 \varepsilon h(x, \dot{x}) = 0 \quad (0 < \varepsilon \ll 1) \]

Examples of \( h \):

1. \( h(x, \dot{x}) = (x^2 - 1) \dot{x} - \text{van der Pol} \)
2. \( h(x, \dot{x}) = \dot{x}^2 - \text{Duffing equation} \)

van der Pol: \( \varepsilon = 0.1 \)

Regular perturbation: failure.

(1) \( x = x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots \)

\[
\begin{align*}
\ddot{x} + 3 \varepsilon \dot{x} + x &= 0 \\
x(0) &= 0, \quad \dot{x}(0) = 1.
\end{align*}
\]

Exact:
\[
x(t, \varepsilon) = \frac{1}{\sqrt{1 - 3\varepsilon^2}} \sin(\sqrt{1 - 3\varepsilon^2} t)
\]

By (1):
\[
\frac{d^2}{dt^2} \left( x_0 + 3x_1 + \ldots \right) + 2 \varepsilon \frac{d}{dt} \left( x_0 + 3x_1 + \ldots \right) + \left( (x_0 + 3x_1 + \ldots) + (x_0 + 3x_1 + \ldots) \right) = 0
\]

\[
\begin{align*}
(x_0 + \varepsilon x_0) + 3 \left( x_1 + 2x_0 + x_1 \right) + O(\varepsilon^2) &= 0 \\
O(1): \quad \ddot{x}_0 + x_0 &= 0 \\
O(\varepsilon): \quad \ddot{x}_1 + 2\dot{x}_0 + x_1 &= 0
\end{align*}
\]

\[
\begin{align*}
x_0(0) &= 0, \quad x_1(0) = 0 \\
x_0(0) &= 1, \quad x_1(0) = 0
\end{align*}
\]

\[
\begin{align*}
\ddot{x}_0 + x_0 &= 0 \\
x_0(0) &= 1, \quad x_0(0) = 0
\end{align*}
\]

\[ \Rightarrow x_0(t) = \sin t \]
\[
\begin{align*}
\dot{x}_1 + x_1 &= -2 \cos t \\
\dot{x}_1(t) &= -x_1(t) \\
\Rightarrow x_1(t) &= -x_1(0) \\
&= 0
\end{align*}
\]

\[
x(t, \varepsilon) = \sin t - 3 t + \sin t + \mathcal{O}(\varepsilon^2)
\]

\[
0(\varepsilon^2 + \varepsilon) \text{ is neglected.}
\]

Method of two-timing:

(1) \[
x(t, \varepsilon) = x_0(T, T) + \varepsilon x_1(T, T) + \mathcal{O}(\varepsilon^2)
\]

- \(T = t\), first time
- \(\tilde{T} = \varepsilon t\), slow time

\[
\dot{x} = \frac{\partial x}{\partial \tilde{T}} T + \frac{\partial x}{\partial T} \frac{\partial T}{\partial \tilde{T}} = \frac{\partial x}{\partial T} + \varepsilon \frac{\partial x}{\partial \tilde{T}}
\]

\[
\left\{ \begin{array}{l}
\dot{x} = \dot{x}_0 + \varepsilon (\dot{x}_1 + \dot{x}_0) + \mathcal{O}(\varepsilon^2)
\end{array} \right.
\]

But \(\dot{x} + 2 \varepsilon \dot{x} + x = 0\).

So, \[
\partial_{\tilde{T}} x_0 + \varepsilon (\partial_{\tilde{T}} x_1 + 2 \partial_T x_0) + 2 \varepsilon \partial_T x_0 + x + \varepsilon x_1 + \mathcal{O}(\varepsilon^2) = 0
\]
(12) \[
\frac{\partial}{\partial t} x_0 + x_0 = 0
\]
(13) \[
\frac{\partial}{\partial t} x_1 + 2 \frac{\partial}{\partial t} x_0 + 2 \varphi x_0 + x_1 = 0
\]
\[
x_0 = A(t) \sin t + \beta(t) \cos t = A(t) \sin t + \beta(t) \cos t
\]
(14) \[
\beta(t) = \frac{\partial}{\partial t} x_0 + \frac{\partial}{\partial t} x_1 = -2 (A^2 + \lambda^2) \cos 2t + 2 (B^2 + \beta^2) \sin t
\]
\[
\begin{align*}
\text{Set } & A^2 + \lambda^2 = 0, & A(t) &= A(0) e^{-t} \\
& B^2 + \beta^2 = 0, & B(t) &= B(0) e^{-t}
\end{align*}
\]
(15) \[
\begin{align*}
0 &= x(0) = x_0(0, 0) + \varepsilon (x_1(0, 0) + o(\varepsilon^2)) \\
& \Rightarrow x_0(0, 0) = 0, \quad x_1(0, 0) = 0
\end{align*}
\]
Similarly,
\[
1 = \frac{\partial}{\partial t} x(0) = \frac{\partial}{\partial t} x_0(0, 0) + \varepsilon \left( \frac{\partial}{\partial t} x_0(0, 0) + \frac{\partial}{\partial t} x_1(0, 0) \right) + o(\varepsilon^2)
\]
\[
\Rightarrow \quad \frac{\partial}{\partial t} x_0(0, 0) = 1,
\]
\[
\frac{\partial}{\partial t} x_0(0, 0) + 2 \varphi x_0(0, 0) = 0
\]
These \( \Rightarrow \beta(0) = 0 \) so \( B(t) \equiv 0 \).
\[
A(0) = 1 \quad \Rightarrow \quad A(t) = e^{-t}
\]
\[
x_0(t, T) = e^{-T} \sin t
\]
\[
\begin{align*}
\chi &= e^{-T} \sin t + o(\varepsilon) \\
&= e^{-\varepsilon t} \sin \theta + o(\varepsilon)
\end{align*}
\]
\[
\varepsilon = 0.1
\]