Part 3. Global Theory

Section 3.1 Concepts: Dynamical Systems. Global Existence, Limit Cycles, and Attractors

Section 3.2 Periodic Structures

Section 3.3 Planar Systems

Section 3.1 Some basic concepts

Consider $\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$

Local existence: $x = x(t)$ $t \in (\alpha, \beta)$ $0 \in (\alpha, \beta)$

max. interval of solution

Remember: $\begin{cases} x' = x^2 \\ x(0) = 1. \end{cases}$

It will be convenient if we can find some equivalent system for which the solution exists for all $t \in (-\infty, \infty)$.

Definition Let $E \neq \phi$ be an open subset of $\mathbb{R}^n$ ~~$[f \in C^1(E, \mathbb{R}^n).]$~~ $\phi \in C^1(\mathbb{R} \times E, E)$ is dynamical system, if $\phi_t(x) = \phi(t, x)$ satisfies

(i) $\phi_0(x) = x$ $\forall x \in E$, and

(ii) $\phi_t \circ \phi_s(x) = \phi_{t+s}(x)$ $\forall s, t \in \mathbb{R}$ $\forall x \in E$.

Remarks

① $\phi_{-t} \circ \phi_t = Id.$

So, $\{\phi_t\}_{t\in\mathbb{R}}$ is a one-parameter family of diffeomorphisms, forming a commutative group under composition.

② $\forall x \in E$. Define

$$f(x) = \frac{d}{dt}\phi(t,x)\Big|_{t=0}.$$

Then $\phi(t, x_0)$ solves $\begin{cases} \dot{x} = f(x), \\ x(0) = x_0. \end{cases}$

Unique solution with the max. interval of existence $I(x_0) = (-\infty, \infty)$.

③ Example Let A: $n\times n$, real matrix, be given.

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0. \end{cases} \Longleftrightarrow x(t) = x_0 e^{At}.$$

So, $\phi(t,x) = xe^{At}$ is a dynamical system on $E = \mathbb{R}^n$.

Definition Let E_1, E_2 be open in $\mathbb{R}^n$. $f \in C^1(E_1, \mathbb{R}^n)$ and $g \in C^1(E_2, \mathbb{R}^n)$. The two systems

$$(1) \; \dot{x} = f(x) \quad \text{and} \quad (2) \; \dot{x} = g(x)$$

are topologically equivalent if $\exists$ homeomorphism $H: E_1 \to E_2$ (one-to-one, onto, continuous) that maps trajectories of (1) to those of (2) and preservs their orientation ~~by time~~ the flows.

More precisely, let $\phi_t(x)$, $\psi_t(x)$ be associated with (1) and (2), respectively. Then, $\forall x \in E_1$,

$\exists\, t = t(x, \tau)$ $(\tau \in \mathbb{R})$ continuously differentiable such that $\frac{\partial t}{\partial \tau} > 0$ $\forall \tau$. (orientation preserving) and

$$H \circ \phi_{t(x,\tau)}(x) = \psi_\tau \circ H(x) \quad \forall x \in E_1 \; \forall \tau \in \mathbb{R}.$$

First talk about Thm on p. 53. Then this Thm.

<u>Theorem</u> (Global existence by rescaling time)
Let $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$. $\forall x_0 \in \mathbb{R}^n$:

$$(*) \quad \begin{cases} \dot{x} = \dfrac{f(x)}{1 + |f(x)|} \\ x(0) = x_0 \end{cases}$$

has a unique solution $x(t)$ <u>defined for all $t \in \mathbb{R}$</u>, i.e., it defines a dynamical system, and this system is topologically equivalent to $\dot{x} = f(x)$.

How the time is rescaled?
Set: $$\tau = \int_0^t \left[1 + |f(x(s))| \right] ds.$$
$$H = Id.$$
Now $$\frac{dx}{d\tau} = \frac{dx}{dt} \Big/ \frac{d\tau}{dt} = \frac{f(x)}{1 + |f(x)|} \quad \text{if } \frac{dx}{dt} = f(x).$$

<u>Proof of Theorem</u> (sketch) $\quad (*) \Leftrightarrow x(t) = x_0 + \int_0^t \frac{f(x(s))\,ds}{1 + |f(x(s))|}$

$\Rightarrow |x(t)| \le |x_0| + \int_0^{|t|} ds = |x_0| + |t| \quad \forall t \in (\alpha, \beta)$: max. interval for solution to $(*)$.

Show $\beta = \infty$ (then same $\alpha = -\infty$). If $\beta < \infty$ then $|x(t)| \le |x_0| + \beta \quad \forall t \in [0, \beta)$. So solution $x(t)$ $(t \in [0,\beta)) \in \underbrace{K = \{x : |x| \le |x_0| + \beta\}}_{\text{compact!}}$

previous result $\Rightarrow$ $\beta = \infty$. <u>Q.E.D.</u>

Example $\begin{cases} x' = x^2 \\ x(0) = x_0 \end{cases}$: $x(t) = \dfrac{x_0}{1 - x_0 t}$ $\begin{cases} t \in (-\infty, \frac{1}{x_0}) \text{ if } x_0 > 0 \\ t \in (\frac{1}{x_0}, \infty) \text{ if } x_0 < 0 \\ t \in (-\infty, \infty) \text{ if } x_0 = 0 \end{cases}$

$$\begin{cases} x' = \dfrac{x^2}{1+x^2} \\ x(0) = x_0 \end{cases}$$

$$2x(t) = t + x_0 - \frac{1}{x_0} + \frac{x_0}{|x_0|}\sqrt{t^2 + 2\left(x_0 - \frac{1}{x_0}\right)t + \left(x_0 + \frac{1}{x_0}\right)^2}$$

$$\begin{cases} \forall t \in (-\infty, \infty) & \text{if } x_0 \neq 0 \\ \equiv 0 & \text{if } x_0 = 0. \end{cases}$$

$$\tau(t) = t + \frac{x_0^2 t}{1 - x_0 t}$$

If $f \in C^1(E, \mathbb{R}^n)$ not $C^1(\mathbb{R}^n, \mathbb{R}^n)$ then the rescaling $\dfrac{f(x)}{1+|f(x)|}$ may not work. check

$\begin{cases} x' = \dfrac{1}{2x} \\ x(0) = x_0. \end{cases}$ the rescaled system is $\begin{cases} x' = \dfrac{1}{2x+1} \\ x(0) = x_0 \end{cases}$

and $I(x_0) = \left(-(x_0 + \frac{1}{2})^2, \infty\right)$ not $\mathbb{R}$.

But, we have

Theorem $\forall f \in C^1(E, \mathbb{R}^n)$ $\exists F \in C^1(E, \mathbb{R}^n)$ s.t.

1. $x' = F(x)$ defines a dynamical system (i.e., solution $x = x(t)$ is defined on $t \in \mathbb{R}$);
2. $x' = F(x)$ and $x' = f(x)$ are topologically equivalent.

Skip the proof.

<u>Theorem</u> If $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the global Lipschitz condition: $\exists K > 0$ s.t.

$$|f(x) - f(y)| \le K|x-y| \quad \forall x, y \in \mathbb{R}^n.$$

then $\forall x_0 \in \mathbb{R}^n$ the system $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ has a unique solution defined on $t \in (-\infty, \infty)$.

<u>Proof</u> Similar idea. First: $\frac{d}{dt}|x(t)| \le |\dot{x}(t)|$

So, $\frac{d}{dt}|x(t) - x_0| \le |\dot{x}(t)| = |f(x(t))|$

$$\le |f(x(t)) - f(x_0)| + |f(x_0)|$$

$$\le K|x(t) - x_0| + |f(x_0)|$$

If the max. sol'n interval is $I(x_0) = (\alpha, \beta)$ with $\beta < \infty$, then, with $g(t) = |x(t) - x_0|$, we have

$$g(t) = \int_0^t \frac{dg(s)}{ds} ds \le |f(x_0)|\beta + K\int_0^t g(s) ds$$

$$\forall t \in (0, \beta).$$

Gronwall's inequality:

$$\Longrightarrow |x(t) - x_0| \le \beta|f(x_0)|e^{K\beta} \quad \forall t \in [0, \beta).$$

Again $\Longrightarrow \beta = \infty$. <u>Q.E.D.</u>

<u>Theorem</u> (Chillingworth) Let M be (an n-dimensional) compact manifold. Then $\forall x_0 \in M$, $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ has a unique solution on $t \in (-\infty, \infty)$. <u>Q.E.D.</u>

Limit Sets and Attractors

Let $E \subseteq \mathbb{R}^n$ be open. Assume $f \in C^1(E, \mathbb{R}^n)$ and

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

has a unique solution $x = x(t) = \phi_t(x_0)$ defined on $t \in (-\infty, \infty)$ i.e., assume $\dot{x} = f(x)$ defines a dynamical system $\phi_t(x)$ $(x \in E,\ t \in \mathbb{R})$.

Notation: $\forall x_0 \in E$

$$\Gamma_{x_0} = \{x \in E : x = \phi(t, x_0),\ t \in \mathbb{R}\}$$
$$= \{\phi(t, x_0) : t \in \mathbb{R}\}.$$

— solution curve or trajectory

Also, $\Gamma_{x_0}^+ = \{\phi(t, x_0) : t \geq 0\}$

$\Gamma_{x_0}^- = \{\phi(t, x_0) : t \leq 0\}$.

$\Gamma = \Gamma^+ \cup \Gamma^-$.

Definition $p \in E$ is an ω-limit (point) of $\phi_t(x) = \phi(t, x)$ if $\exists\, t_n \uparrow \infty$ s.t.

$$\lim_{n\to\infty} \phi(t_n, x) = p.$$

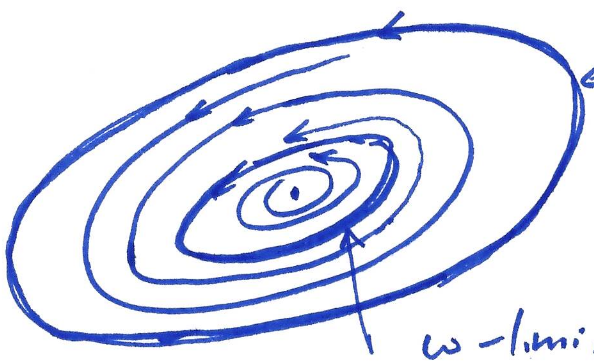
$q \in E$ is an α-limit of $\phi(t, x)$ if $\exists\, t_n \downarrow -\infty$ s.t.

$$\lim_{n\to\infty} \phi(t_n, x) = q.$$

$\omega(\Gamma) = \omega$-limit set of Γ
$= \{\omega\text{-limit points of } \phi(t, x),\ x \in \Gamma\}$

$\alpha(\Gamma) = \alpha$-limit set of Γ
$= \{\alpha\text{-limit points of } \phi(t, x),\ x \in \Gamma\}$

$\omega(\Gamma) \cup \alpha(\Gamma)$: limit set of Γ



<u>Theorem</u> ① $\alpha(\Gamma)$, $\omega(\Gamma)$ are closed in E.
② If Γ is bounded then $\alpha(\Gamma)$, $\omega(\Gamma)$ are nonempty, connected, and compact subsets of E.

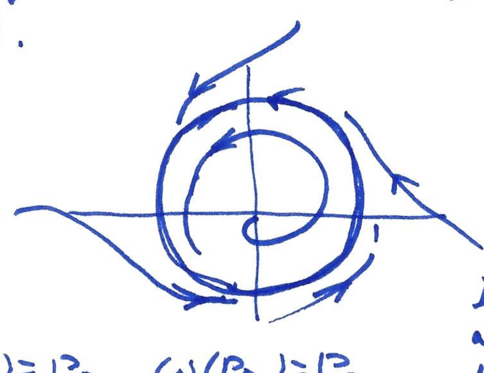
<u>Theorem</u> $p \in \omega(\Gamma) \Longrightarrow \Gamma_p \subseteq \omega(\Gamma)$.
$p \in \alpha(\Gamma) \Longrightarrow \Gamma_p \subseteq \alpha(\Gamma)$.

In particular, $\alpha(\Gamma)$, $\omega(\Gamma)$ are invariant w.r.t. the flow ϕ_t defined by $\dot{x} = f(x)$: $\underline{\phi_t(\alpha(\Gamma)) \subseteq \alpha(\Gamma)}$, $\underline{\phi_t(\omega(\Gamma)) \subseteq \omega(\Gamma))}$.

This one is intuitively true; we skip the pf.

$A \subseteq E$ is <u>invariant</u> if $\phi_t(A) \subseteq A$.
<u>Attracting set</u> of $\dot{x} = f(x)$: closed invariant set $A \subseteq E$ s.t. $\exists$ a neighborhood of A, call it U, s.t.
$x \in U \Longrightarrow \phi_t(x) \in U \ \ \forall t \geq 0$ and $x(t) \to A$ as $t \to \infty$.
An <u>attractor</u>: an attracting set containing a dense orbit (trajectory).

<u>Example</u> $\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 \end{cases}$



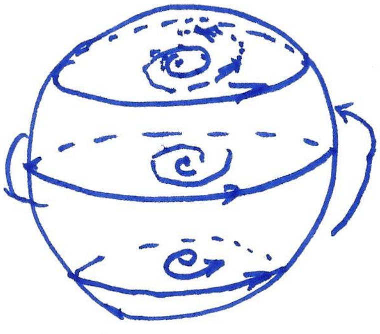
Let $\Gamma_0 \equiv$ counterclockwise flow on unit circle

Γ_0: an attractor. $\alpha(\Gamma_0) = \Gamma_0$. $\omega(\Gamma_0) = \Gamma_0$.

In fact, Γ_0 is a stable limit cycle.

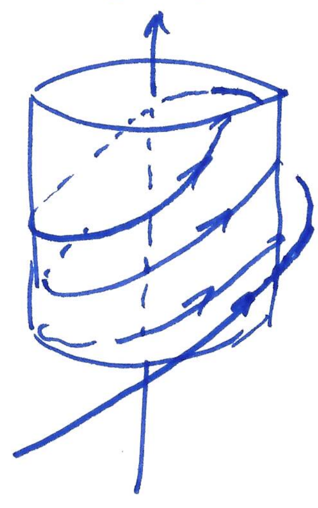
Example
$$\begin{cases} \dot{x} = -y + x(1-x^2-y^2-z^2) \\ \dot{y} = x + y(1-x^2-y^2-z^2) \\ \dot{z} = 0 \end{cases}$$

$\dot{z}=0$: planar motion.
So, $\{z = \text{const}\}$: invariant.
$S^2 \cup \{(0,0,z) : |z| \geq 1\}$
is an attracting set.



Example
$$\dot{x} = -y + x(1-x^2-y^2)$$
$$\dot{y} = x + y(1-x^2-y^2)$$
$$\dot{z} = \alpha \qquad (\alpha = \text{const.} \neq 0)$$

z-axis $\cup$ cylinder $x^2+y^2=1$ is invariant.
the cylinder is an attracting set.



Γ

Example An invariant torus as an attracting set.

Example The Lorenz system.
$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = \rho x - y - xz \\ \dot{z} = -\beta z + xy \end{cases}$$
Google search this system!

Example 1-d. $\dot{x} = -x^4 \sin\left(\frac{\pi}{x}\right)$. critical pts: 0, $\pm\frac{1}{n}$, $n=1,2,\cdots$
$[-1,1]$: an attracting set. $\pm\frac{1}{2n}$ $(n=1,2,\cdots)$: repelling.
$\pm\frac{1}{2n-1}$ $(n=1,2,\cdots)$: attracting.

Section 2 Periodic Structures: Periodic Orbits, Limit Cycles, and Separatrix Cycles

Notation: $E \subseteq \mathbb{R}^n$ open. $f \in C^1(E, \mathbb{R}^n)$. $\phi(t,x)$ is the dynamic system defined by $\dot{x} = f(x)$; .....(1)

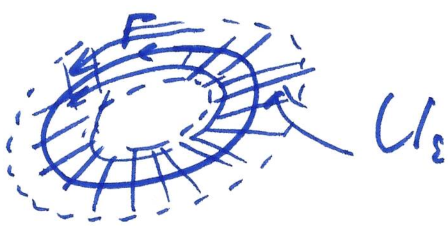
<u>Definition</u> A <u>closed orbit</u> or <u>cycle</u> of (1) is a closed solution curve of (1) that is not an equilibrium point of (1).

For a cycle Γ

<u>Stable</u>: If for any $\varepsilon > 0$, $\exists$ a neighborhood U_ε of Γ s.t.

$$x \in U_\varepsilon \Rightarrow dist(\phi_t(x), \Gamma) < \varepsilon \quad \forall t \geq 0$$

<u>unstable</u>: If not stable.



<u>Asymptotically stable</u>

If it is stable and

$\exists$ n.b.h. U of Γ s.t. $x \in U \Rightarrow \lim_{t \to \infty} dist(\phi_t(x), \Gamma) = 0$.

<u>Proposition</u>. Any asymptotically stable cycle is an attractor.
Q.E.D.

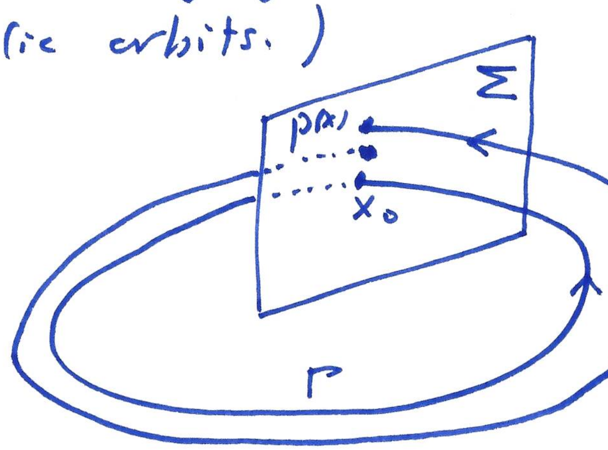
Talk about limit cycles first. Cf. p.60.

<u>The Poincaré Map</u> (or first return map)

(A tool for studying the stability and bifurcations of periodic orbits.)

$x \longmapsto P(x)$

$\Sigma \longmapsto \Sigma$



Γ: closed orbit

$x_0 \in \Gamma$. hyperplane

$\Sigma \perp \Gamma$ at x_0.

x near x_0

$x \in \Sigma$.

$\Rightarrow$ 1st return $P(x) \in \Sigma$.

Theorem Let $x_0 \in E$. Suppose $\phi_t(x_0)$ is T-periodic and $\Gamma = \{x \in \mathbb{R}^n : x = \phi_t(x_0),\ 0 \le t \le T\} \subseteq E$. Let $\Sigma = \{x \in \mathbb{R}^n : (x - x_0)\cdot f(x_0) = 0$. [i.e., Σ is a hyperplane in $\mathbb{R}^n$ that is perpendicular to Γ at x_0.].

Then $\exists\, \delta > 0$, and a unique $\tau \in C^1(B(x_0,\delta))$ s.t. $\tau(x_0) = T$ and $\phi_{\tau(x)}(x) \in \Sigma \quad \forall x \in B(x_0,\delta)$.

The map $P(x) = \phi_{\tau(x)}(x)$, $\forall x \in B(x_0,\delta) \cap \Sigma$, is called the Poincaré map for Γ at x_0.

Proof Use the Implicit Function Theorem (IFM).
Define $F(t,x) = [\phi_t(x) - x_0]\cdot f(x_0)$. $F \in C^1(\mathbb{R}\times E)$, and $F(T, x_0) = 0$.
Since $\phi(t, x_0) = \phi_t(x_0)$ solves $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ and $\phi(T, x_0) = x_0$, we have

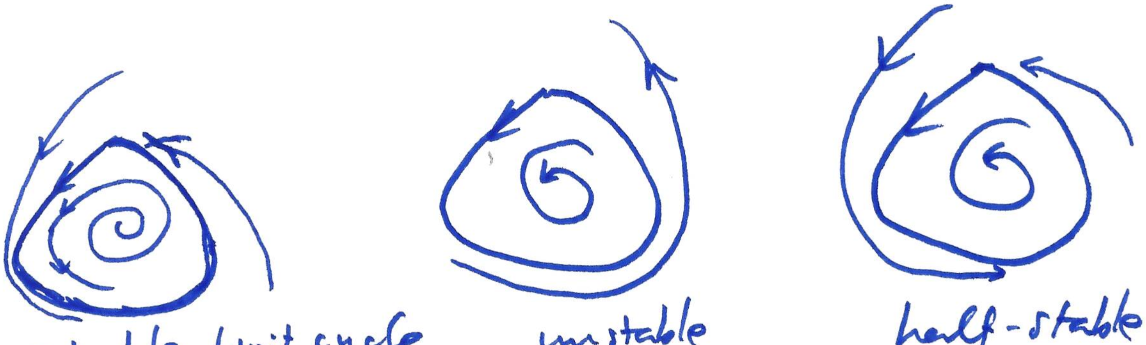
$$\frac{\partial F(T,x_0)}{\partial t} = \frac{\partial \phi(T,x_0)}{\partial t}\cdot f(x_0) = f(x_0)\cdot f(x_0) = |f(x_0)|^2 \neq 0$$

since x_0 is not an equilibrium point.

Now, IFM $\Longrightarrow \exists\, \delta > 0$, $\exists!\ \tau \in C^1(B(x_0,\delta))$ such that $\tau(x_0) = T$ and $F(\tau(x), x) = 0 \quad \forall x \in B(x_0,\delta)$. Hence, for any $x \in B(x_0,\delta)$, $[\phi(\tau(x), x) - x_0]\cdot f(x_0) = 0$, i.e. $\phi_{\tau(x)}(x) \in \Sigma$. Q.E.D.

We now present one example of computing the Poincaré maps. We then describe the idea of using the Poincaré map to study the stability of a cycle in $\mathbb{R}^2$. After that, we state the general stability results for $\mathbb{R}^n$.
First, let's introduce the concept of limit cycles (in $\mathbb{R}^2$)

A limit cycle is an isolated closed trajectory. Here, isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle.



stable limit cycle — unstable limit cycle — half-stable limit cycle

$\Updownarrow$

Same as asymptotically stable limit cycle.
$\Longrightarrow$ an attractor!

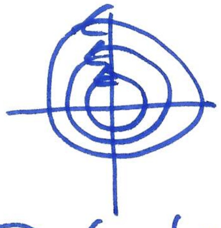
] Describe self-sustained oscillations.

Examples of stable limit cycles: heart beating; periodic firing of a pace maker neuron, daily rhythms of human body temperatures and hormone secretion; and dangerous self-excited vibrations in bridges and airplane wings.

<u>Proposition</u> ① Linear systems have no limit cycles.

$\dot{x} = Ax$: periodic solutions are not isolated

$x = x(t)$ is periodic
$\Longrightarrow$ $x = c\,x(t)$ is also periodic.

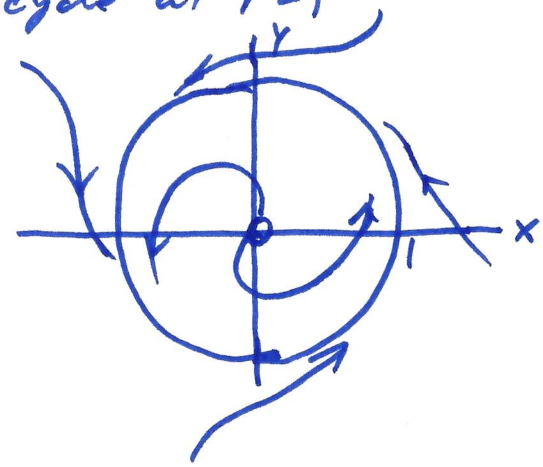


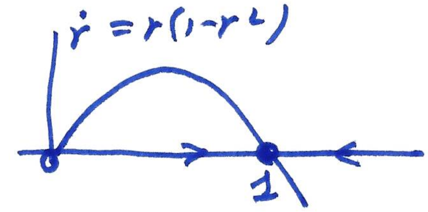
So, limit cycles are nonlinear phenomena.

② Gradient systems do not permit limit cycles.
$\dot{x} = -\nabla U(x)$.

<u>Example</u> $\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \end{cases} \Longleftrightarrow \begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 \end{cases}$

Limit cycle at $r=1$





Construct the Poincaré map for the limit cycle $r=1$. P: $T=2\pi$.

at $(r(0), \theta(0)) = (r_0, \theta_0)$.

First solve $\dot{r} = r(1-r^2)$ to get

$$r(t, r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-2t}\right]^{-1/2}$$

solve $\dot{\theta} = 1$ to get

$$\theta(t, \theta_0) = t + \theta_0$$

So, for $\Sigma: \theta = \theta_0$. s.t. $(r_0, \theta_0) \in \Sigma \cap \Gamma$ at $t=0$. intersects $\theta = \theta_0$ again at $t = 2\pi$. Thus,

$$P(r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}\right]^{-1/2}$$ — The Poincaré map.

In particular, $P(1) = 1$.

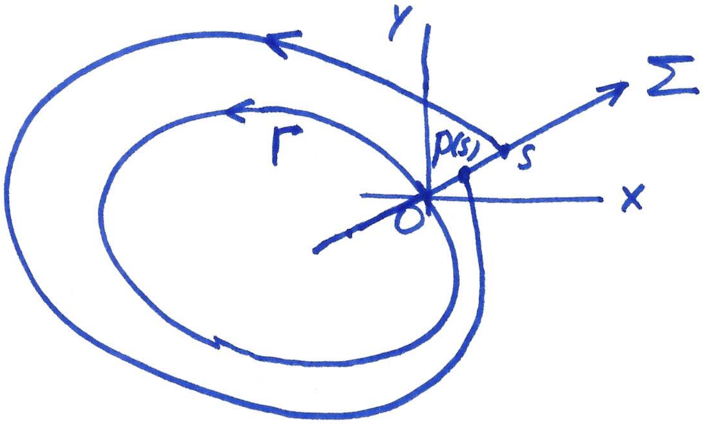
Note that

$$P'(r_0) = e^{-4\pi} r_0^{-3}\left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}\right]^{-3/2}$$

$$P'(1) = e^{-4\pi} < 1.$$

This in fact implies the stability.

Consider $\mathbb{R}^2$. $x_0 = 0 \in \Gamma \cap \Sigma$. Γ: cycle. $\Sigma \perp \Gamma$ at $x_0 = 0$.



$\Sigma = \Sigma^- \cup \{0\} \cup \Sigma^+$

$s > 0$ in Σ^+

$s < 0$ in Σ^-

$P(0) = 0$.

P: the Poincaré map of Γ at 0. defined on $[-\delta, \delta]$

Introduce the displacement function

$$d(s) = P(s) - s.$$

Then $d'(s) = P'(s) - 1$.

The Mean-Value Thm $\Rightarrow \exists\, \sigma$ in between $0, s$:

$$d(s) = d'(\sigma)\, s \qquad [\, d(s) - d(0) = d'(\sigma)(s-0) \,]$$

Suppose $d'(0) = P'(0) - 1 \neq 0$. Then

$$d(s) = d'(0)\, s + o(s) \quad \text{if } |s| \ll 1$$

$$\left[\begin{aligned} d(s) &= d'(\sigma)\, s = [d'(\sigma) - \cancel{1}^{\,d'(0)}]\, s + \cancel{s}\, d'(0) s \\ &= d'(0) s + o(s) \quad \text{as } |\sigma| \le |s| \ll 1. \end{aligned} \right]$$

Thus

$$d'(0) < 0 \Rightarrow \begin{cases} d(s) < 0 & \text{for } s > 0 \\ d(s) > 0 & \text{for } s < 0 \end{cases} \Rightarrow \Gamma \text{ is stable}$$

$$d'(0) > 0 \Rightarrow \begin{cases} d(s) < 0 & \text{for } s < 0 \\ d(s) > 0 & \text{for } s > 0 \end{cases} \Rightarrow \Gamma \text{ is unstable.}$$

A beautiful idea!

A remark: if $d(0) = d'(0) = \cdots = d^{(k-1)}(0) = 0$.

$d^{(k)}(0) \neq 0$.

Then $d^{(k)}(0) < 0 \Rightarrow$ stable
$d^{(k)}(0) > 0 \Rightarrow$ unstable.] If k is odd.

semi stable if k is even.

<u>Theorem</u> Let $\gamma(t)$ be a periodic solution of $\dot{x}=f(x)$ of period T. ~~Bi~~ Let $P(s)$ be the Poincaré map along a straight line Σ normal to $\Gamma=\{x\in\mathbb{R}^2: x=\gamma(t)-\gamma(0), 0\le t\le T\}$ at $x=0$. Then

$$P'(0)=\exp\int_0^T \nabla\cdot f(\gamma(t))\,dt.$$

Moreover

$$\int_0^T \nabla\cdot f(\gamma(t))\,dt \begin{cases} <0 \Rightarrow \gamma(t) \text{ is a stable limit cycle.} \\ >0 \Rightarrow \gamma(t) \text{ is an unstable limit cycle.} \\ =0 \Rightarrow \gamma(t) \text{ may be a stable, unstable,} \end{cases}$$

or semi-stable limit cycle or it may belong to a continuous band of cycles.

<u>Example</u> $\begin{cases} \dot{r}=r(1-r^2) \\ \dot{\theta}=1. \end{cases}$ $\gamma(t)=\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}$, $T=2\pi$

or $\begin{cases} \dot{x}=-y+x(1-x^2-y^2) \\ \dot{y}=x+y(1-x^2-y^2) \end{cases}$

$$\nabla\cdot f(x,y)=2-4x^2-4y^2$$

$$\int_0^{2\pi}\nabla\cdot f(\gamma(t))\,dt=\int_0^{2\pi}(2-4\cos^2 t-4\sin^2 t)\,dt=-4\pi<0.$$

$\Rightarrow P'(0)=\cancel{-4\pi}\, e^{-4\pi}<1.$ $\gamma(t)$: stable limit cycle.

For general $\mathbb{R}^n$, $\dot{x}=f(x)$, suppose Γ: $\gamma(t)$ is a periodic orbit with period T. Let P be the Poincaré map of Γ at x_0. Then $DP(x_0)$ is $(n-1)\times(n-1)$ matrix. If $\|DP(x_0)\|<1$ then Γ is asymptotically stable. If P is given by $\phi_t(x)=\phi(t,x)$, $x_0=0\in\Gamma$. the x_n-axis is tangent to Γ at 0, pointing same direction as the motion along Γ, then $DP(0)=\left[\frac{\partial\phi_i}{\partial x_j}(T,0)\right]_{i,j=1}^{n-1}$.

The $n-1$ eigenvalues of $DP(0)$ are $e^{\lambda_1 T}, \dots, e^{\lambda_{n-1} T}$

$0, \lambda_1, \dots, \lambda_{n-1}$ are eigenvalues of a constant $n \times n$ matrix B:

$$A(t) = Df(\gamma(t)) = Df(\phi_t(x)) \qquad \gamma(t): \text{ cycle of period } T$$

$$\dot{x} = A(t)x$$

has the fundamental matrix $\Phi(t)$ defined by

$$\begin{cases} \Phi'(t) = A(t)\Phi(t) \\ \Phi(0) = I \end{cases}$$

Floquet's Theorem $\Rightarrow$ $\Phi(t) = \underset{\substack{\uparrow \\ \text{periodic}}}{Q(t)} e^{Bt}$ This defines B.

Eigenvalues of B are: $0, \lambda_1, \dots, \lambda_{n-1}$. These define stable, unstable, and center manifolds of P; they are invariant sets of $\phi_t(x)$.

A general result. $E \subseteq \mathbb{R}^n$ open. $f \in C^1(E, \mathbb{R}^n)$. $\dot{x} = f(x)$ defines a dynamical system $\phi(t, x) = \phi_t(x)$.

<u>Theorem</u> If $\gamma(t)$ is a periodic orbit with period T, then $\int_0^T \nabla \cdot f(\gamma(t))\,dt > 0 \Rightarrow \gamma(t)$ is not asymptotically stable.

<u>Remark</u> For $n \geq 3$, $\int_0^T \nabla \cdot f(\gamma(t))\,dt < 0$ does not imply that $\gamma(t)$ is asymptotically stable.

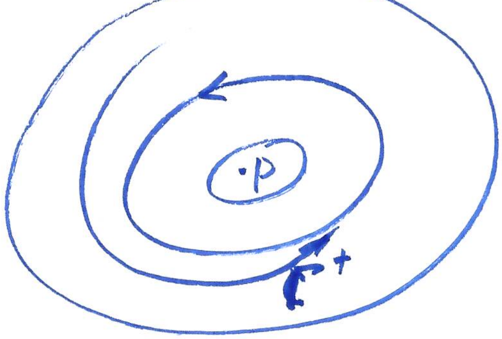
Section 3. Planar Systems : Stability of Periodic Solutions, applications, etc.

In this section, we assume:

- $E \subseteq \mathbb{R}^2$: open (nonempty)
- $f \in C^1(E, \mathbb{R}^2)$
- $\dot{x} = f(x)$ defines a dynamical system $\varphi_t(x) = \varphi(t, x) : \mathbb{R} \times E \to E$, $\varphi \in C^1$.

Theorem (Poincaré-Bendixson) Let Γ be a trajectory of $\dot{x} = f(x)$ with Γ^+ contained in a compact subset K of E. Then $\omega(\Gamma)$ contains no critical point of $\dot{x} = f(x)$. Moreover, $\omega(\Gamma)$ is a periodic orbit of $\dot{x} = f(x)$.

A different version of Poincaré-Bendixson Theorem.



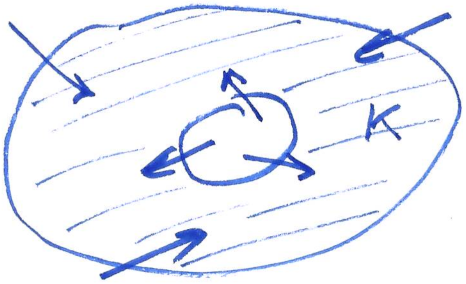
Assume

1. $\exists$ a compact set $K \subset E$, that does not contain any critical points of f.
2. $\exists$ a trajectory Γ confined in K.

[~~in K~~, i.e. $\forall x \in \Gamma \subseteq K \Longrightarrow \varphi_t(x) \in K \quad \forall t \geq 0$]

Then, Γ is a closed orbit or it spirals toward a closed orbit as $t \to \infty$.

The trapping region method.



K: compact connected
on the boundary ∂K.
flow in.
$\Longrightarrow$ all trajectories are confined in K.

If $\exists$ no critical points in K, then K contains a closed orbit.

<u>Example</u> $\begin{cases} \dot{r} = r(1-r^2) + \mu r\cos\theta \\ \dot{\theta} = 1 \end{cases}$

$\mu = 0$: stable limit cycle at $r=1$.
We show now that $0 < \mu \ll 1 \Longrightarrow$ a closed orbit exists

Solution: We find r_{min}, r_{max} : $0 < r_{min} < r_{max} < \infty$.
We shall consider $K = \{r : r_{min} \le r \le r_{max}\}$
with $\dot{r} < 0$ at $r = r_{max}$, $\dot{r} > 0$ at $r = r_{min}$, and
K has no critical points of the system.

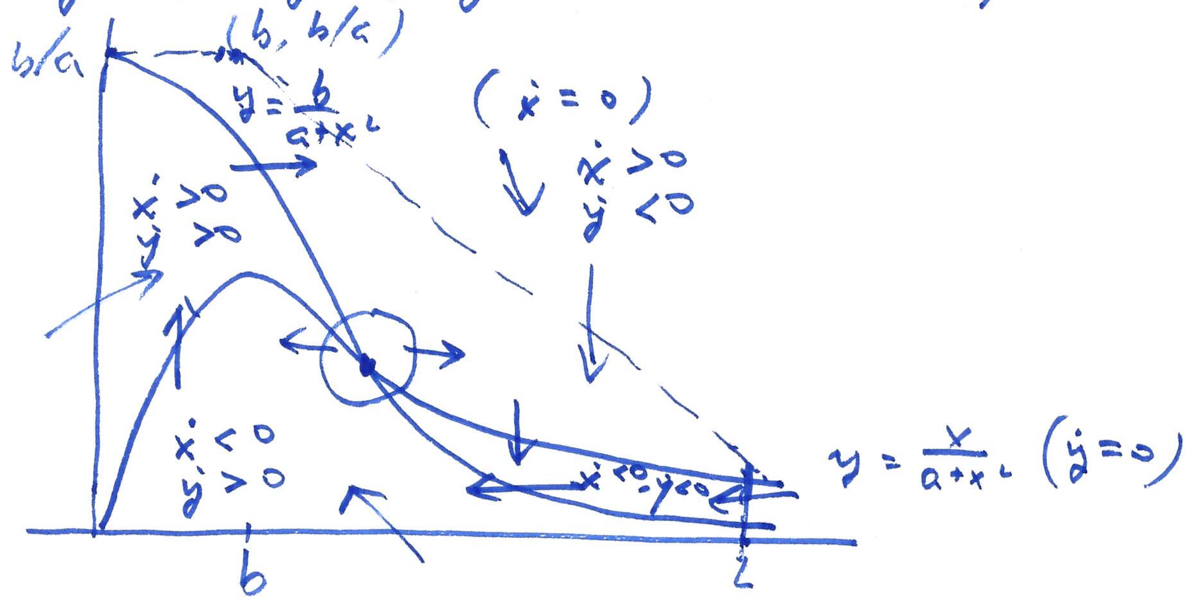
For r_{min}: require $\dot{r} = r(1-r^2) + \mu r\cos\theta > 0$
for all θ. $r(1-r^2) + \mu r\cos\theta \ge r(1-r^2) - \mu r > 0$
$1 - r^2 - \mu > 0$, $\boxed{r_{min} < \sqrt{1-\mu}}$ $(0 < \mu < 1)$

similarly. $\boxed{r_{max} > \sqrt{1+\mu}.}$

e.g. $r_{min} = 0.999\sqrt{1-\mu}$, $r_{max} = 1.001\sqrt{1+\mu}$.
In $\{r_{min} < r < r_{max}\}$, no crt pts. So, a closed orbit exists.

<u>Example</u> A model for glycolysis – a ~~α~~ biochemical process.
x = ADP concentration
y = F6P concentration

$$\begin{cases} \dot{x} = -x + ay + x^2 y \\ \dot{y} = b - ay - x^2 y \end{cases} \qquad (a, b > 0)$$



$K =$  $- \bigcirc$.

The small disk contains a crt. pt.

Liénard Systems

(1) $\quad \ddot{x} + f(x)\dot{x} + g(x) = 0$

Generalization of the van der Pol oscillator

(2) $\quad \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$

Rewrite (1):
$$(3) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -g(x) - f(x)\, y. \end{cases}$$

<u>Liénard's Theorem</u> Suppose

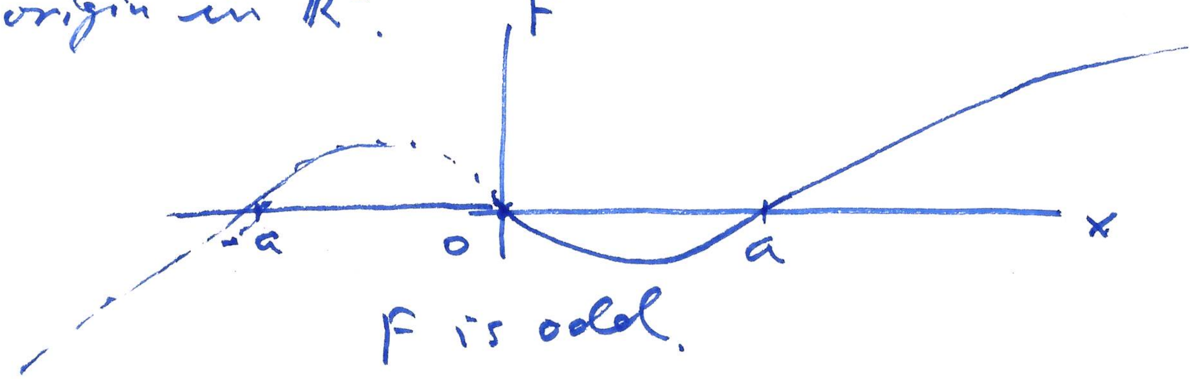
(1) $f, g \in C^1$

(2) f is even: $f(-x) = f(x)$

(3) g is odd: $g(-x) = -g(x)$ and $g(x) > 0 \ \forall x > 0$.

(4) $F(x) = \int_0^x f(u)\,du$ has exactly one positive zero $a > 0$, negative in $0 < x < a$, positive and nondecreasing for $x > a$, and $\lim_{x \to +\infty} F(x) = \infty$.

Then (3) has a unique, stable limit cycle surrounding the origin in $\mathbb{R}^2$.



F is odd.

<u>Example</u>. For the van der Pol equation

$f(x) = \mu(x^2 - 1)$. $g(x) = x$.

$F(x) = \mu\left(\frac{1}{3}x^3 - x\right) = \frac{1}{3}\mu x(x^2 - 3)$. $a = \sqrt{3}$.

So, Thm $\Longrightarrow \exists$ unique, stable limit cycle.

We now analyze ~~to~~ the van der Pol system for $\mu \gg 1$.

$$\ddot{x} + \mu\dot{x}(x^2 - 1) = \frac{d}{dt}\left(\dot{x} + \mu\left(\frac{1}{3}x^3 - x\right)\right)$$

Let $F(x) = \frac{1}{3}x^3 - x$, $w = \dot{x} + \mu F(x)$

$$\Longrightarrow \quad \dot{w} = \ddot{x} + \mu\dot{x}(x^2 - 1) = -x$$

So, equivalently $\begin{cases} \dot{x} = w - \mu F(x) \\ \dot{w} = -x \end{cases}$

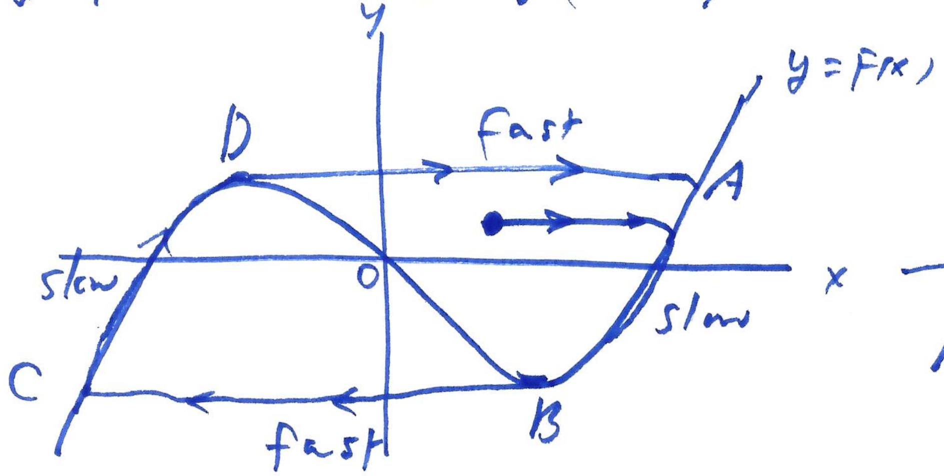
$\dot{x} = P(x, y)$
$\dot{y} = Q(x, y)$

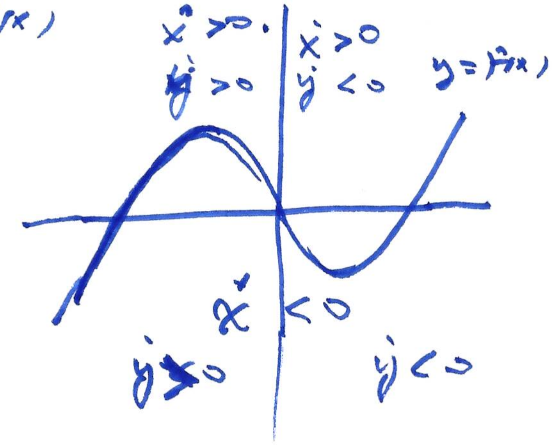
Nullclines: curves
$P(x, y) = 0$.
$Q(x, y) = 0$.

Change variable: $y = \frac{w}{\mu}$.

$$\begin{cases} \dot{x} = \mu\,[y - F(x)] \\ \dot{y} = -\frac{1}{\mu} x \end{cases}$$

$y - F(x) = 0 \iff y = \frac{1}{3}(x^3 - x)$





$\mu \gg 1$: fast horizontal motion
slow vertical motion.

Estimate the period of the limit cycle for $\mu \gg 1$
The period $T \approx$ time for travelling along the two slow branches. $\approx 2\int_{t_A}^{t_B} dt$

$y \approx F(x)$

$\dot{y} \approx F'(x)\,\dot{x} = (x^2 - 1)\,\dot{x}$

But, $\dot{y} = -\frac{x}{\mu}$.

So, $\dot{x} \approx -\frac{x}{\mu(x^2 - 1)}$

$\frac{dx}{dt} \approx -\frac{x}{\mu(x^2-1)}$. $\quad dt \approx -\frac{\mu(x^2-1)}{x}\,dx$

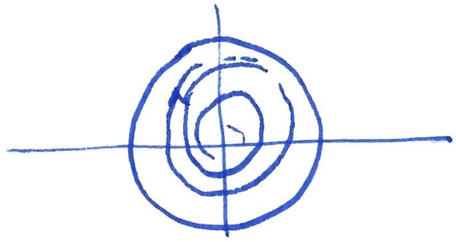
$x_A = 2$, $x_B = 1$

$$T = 2\int_2^1 -\frac{\mu}{x}(x^2 - 1)\,dx = 2\mu\left[\frac{x^2}{2} - \ln x\right]_1^2$$
$$= \mu\,(3 - 2\ln 2) = O(\mu).$$

Weekly nonlinear oscillators

$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0 \qquad (0 < \varepsilon \ll 1)$

Examples of h: ① $h(x,\dot{x}) = (x^2-1)\dot{x}$ — van der Pol
② $h(x,\dot{x}) = x^3$ — Duffing equation.



van der Pol. $\varepsilon = 0.1$

Regular perturbation: failure.

(1) $x = x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$

$$\begin{cases} \ddot{x} + 2\varepsilon\dot{x} + x = 0 \\ x(0) = 0,\ \dot{x} = 1. \end{cases}$$

Exact:
$$x(t,\varepsilon) = \frac{1}{\sqrt{1-\varepsilon^2}} e^{-\varepsilon t} \sin\left(\sqrt{1-\varepsilon^2}\, t\right)$$

By (1) $\frac{d^2}{dt^2}(x_0 + \varepsilon x_1 + \cdots) + 2\varepsilon \frac{d}{dt}(x_0 + \varepsilon x_1 + \cdots) + (x_0 + \varepsilon x_1 + \cdots) = 0$

$$(\ddot{x}_0 + x_0) + \varepsilon(\ddot{x}_1 + 2\dot{x}_0 + x_1) + O(\varepsilon^2) = 0$$

$O(1)$: $\ddot{x}_0 + x_0 = 0$

$O(\varepsilon)$: $\ddot{x}_1 + 2\dot{x}_0 + x_1 = 0$

$x_0(0) = 0 \quad x_1(0) = 0$

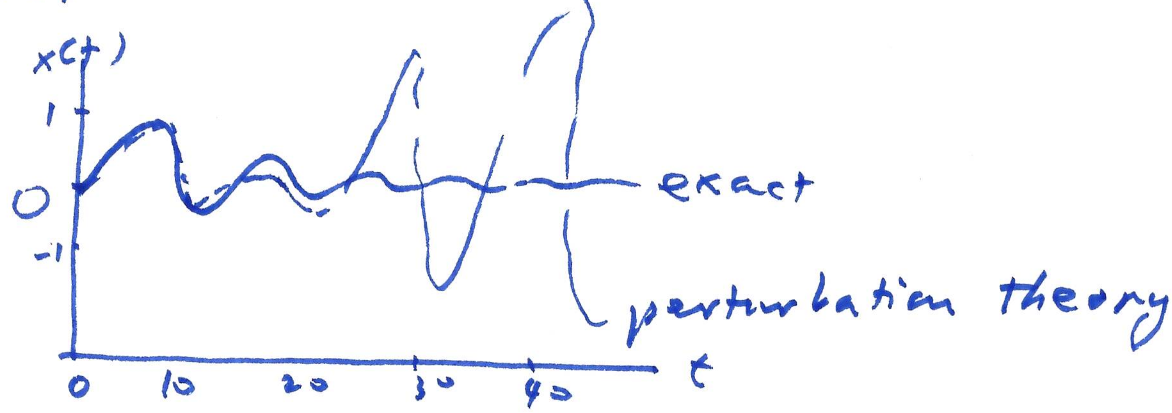
$\dot{x}_0(0) = 1 \quad \dot{x}_1(0) = 0$

$$\begin{cases} \ddot{x}_0 + x_0 = 0 \\ x_0(0) = 0,\ \dot{x}_0(0) = 1 \end{cases} \Longrightarrow \quad x_0(t) = \sin t$$

$$\begin{cases} \ddot{x}_1 + x_1 = -2\cos t \\ x_1(0)=0 \quad \dot{x}_1(0)=0 \end{cases} \Longrightarrow x_1(t) = -t\sin t$$

[bad! non periodic, grow as $t \to \infty$]

$$x(t,\varepsilon) = \sin t - \varepsilon t \sin t + O(\varepsilon^2)$$



$O(\varepsilon^2 t^2)$ is neglected.

Method of two-timing.

(1) $x(t,\varepsilon) = x_0(\tau,T) + \varepsilon x_1(\tau,T) + O(\varepsilon^2)$

$\tau = t$, — fast time

$T = \varepsilon t$. — slow time

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial x}{\partial \tau} + \varepsilon\frac{\partial x}{\partial T}$$

[$\dot{x} = \partial_\tau x + \varepsilon \partial_T x$]

(1)$\Rightarrow$ $\dot{x} = \partial_\tau x_0 + \varepsilon(\partial_T x_0 + \partial_\tau x_1) + O(\varepsilon^2)$

$\ddot{x} = \partial_{\tau\tau} x_0 + \varepsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + O(\varepsilon^2)$.

But $\ddot{x} + 2\varepsilon\dot{x} + x = 0$.

So, $\partial_{\tau\tau} x_0 + \varepsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + 2\varepsilon\partial_\tau x_0 + x_0 + \varepsilon x_1 + O(\varepsilon^2) = 0$

$O(1)$: $\partial_{\tau\tau} x_0 + x_0 = 0$ (2)

$O(\varepsilon)$: $\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0 + 2\partial_\tau x_0 + x_1 = 0$ (3)

$x_0 = A \sin\tau + B\cos\tau = A(T)\sin\tau + B(T)\cos\tau$ (4)

Put x_0 into (3):

$$\partial_{\tau\tau} x_1 + x_1 = -(2\partial_{T\tau} x_0 + \partial_\tau x_0)$$
$$= -2(A'+A)\cos\tau + 2(B'+B)\sin\tau.$$

Set $A'+A=0$ $\quad A(T)=A(0)e^{-T}$

$B'+B=0$ $\quad B(T)=B(0)e^{-T}$

Finally, find $A(0), B(0)$, by (1) and (2) and $x(0)=0$, $\dot{x}(0)=1$.

$$0 = x(0) = x_0(0,0) + \varepsilon\, x_1(0,0) + O(\varepsilon^2).$$
$$\Longrightarrow x_0(0,0)=0, \quad x_1(0,0)=0$$

Similarly,

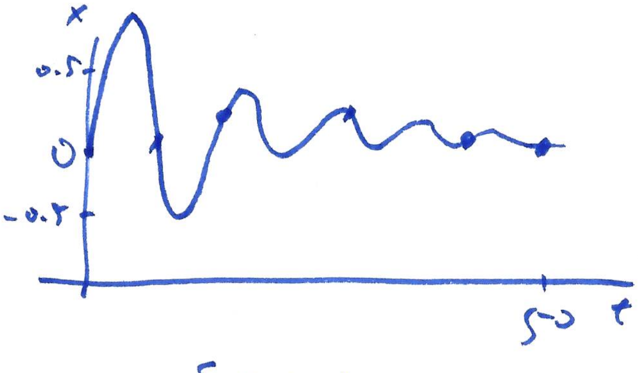
$$1 = \dot{x}(0) = \partial_\tau x_0(0,0) + \varepsilon\left(\partial_T x_0(0,0) + \partial_\tau x_1(0,0)\right) + O(\varepsilon^2)$$
$$\Longrightarrow \partial_\tau x_0(0,0) = 1.$$
$$\partial_T x_0(0,0) + \partial_\tau x_1(0,0) = 0$$

These $\to$ $B(0)=0$ so $B(T)\equiv 0$.

$A(0)=1 \Longrightarrow A(T)=e^{-T}$

$x_0(\tau,T) = e^{-T}\sin\tau$

$$\boxed{\begin{aligned} x &= e^{-T}\sin\tau + O(\varepsilon) \\ &= e^{-\varepsilon t}\sin t + O(\varepsilon) \end{aligned}}$$



$\varepsilon = 0.1$