

Math 210 C. Winter 2018 (B. Li; UCSD)

Soln to HW #5 Problems

$$1. |z| = p, \quad z = pe^{i\theta} \quad (0 \leq \theta < 2\pi)$$

$$z^{-1} = p^{-1}e^{-i\theta}$$

$$w = u + iv = z + \frac{1}{z} = pe^{i\theta} + p^{-1}e^{-i\theta}$$

$$= p \cos \theta + i p \sin \theta + p^{-1} \cos \theta - i p^{-1} \sin \theta$$

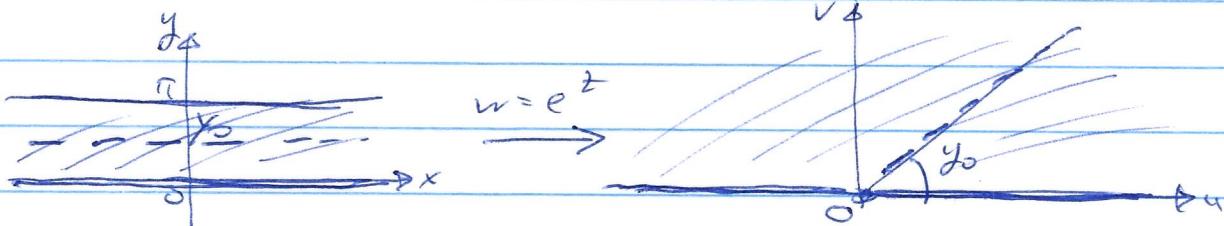
$$(P+P^{-1}) \cos \theta + i(p-p^{-1}) \sin \theta$$

Hence $u = (p+p^{-1}) \cos \theta$ $(0 \leq \theta < 2\pi)$
 $v = (p-p^{-1}) \sin \theta$

and

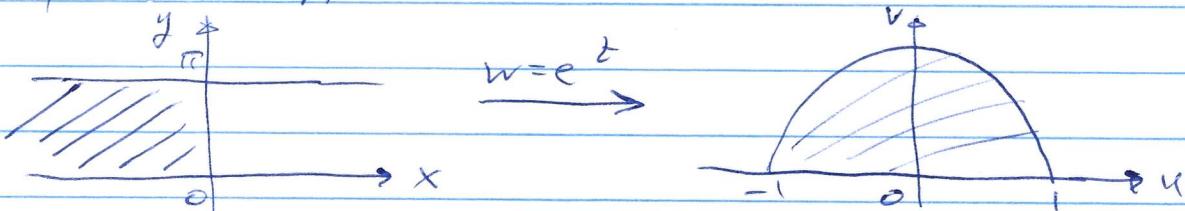
$$\left(\frac{u}{p+p^{-1}}\right)^2 + \left(\frac{v}{p-p^{-1}}\right)^2 = 1.$$

$$2. (1) w = e^z = e^x e^{iy}. \quad y=0: w = e^x, \quad y=\pi \quad w = -e^{-x}$$

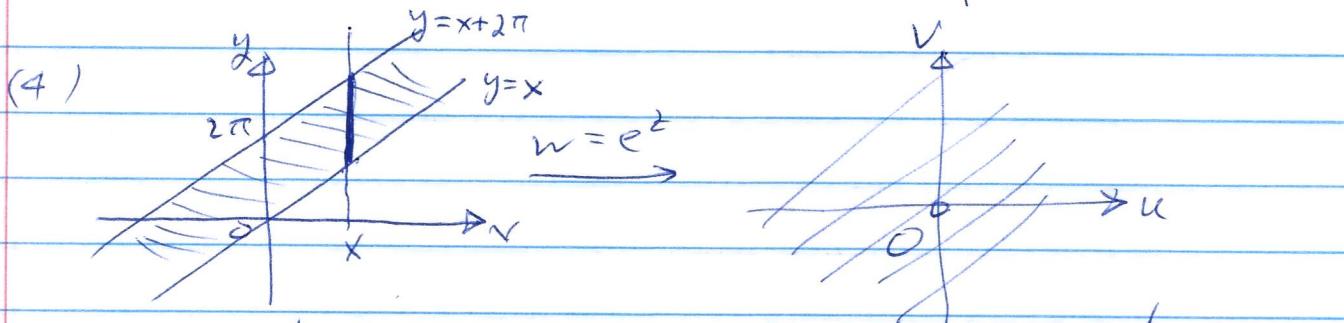
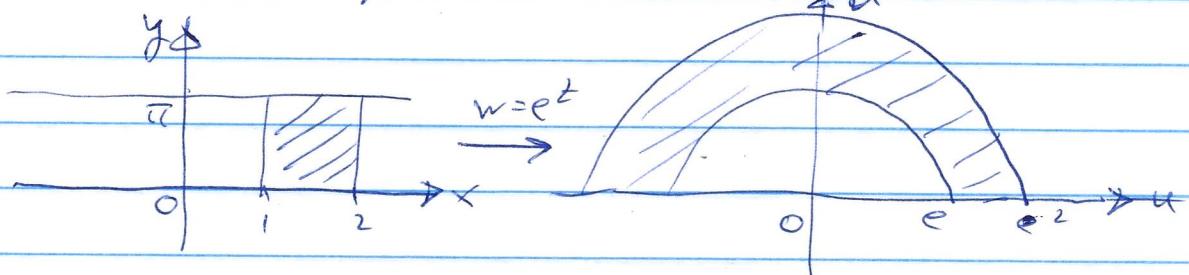


Each ray $0 < \operatorname{Im} z = y_0 < \pi$ is mapped by $w = e^z$ to half line $\theta = y_0$ in w -plane. So, $0 < \operatorname{Im} z < \pi$ is mapped to $v > 0$, upper half plane (open) (not including the u -axis).

(2) $\operatorname{Re} z < 0, 0 < \operatorname{Im} z < \pi$ is mapped by $w = e^z$ to the upper half unit disk $|w| < 1, u > 0$.



(3) $1 < \operatorname{Re} z < 2$, $0 < \operatorname{Im} z < \pi$ is mapped by $w = e^z$ to the ^{upper half} annulus $e < |w| < e^2$, $u > 0$.



This is mapped to the entire w -plane minus $\{0\}$.

Reason: $\forall x \in \mathbb{R}$, $w = e^x \cdot e^{iy}$ ($y \in (-\pi, \pi]$) is a circle of radius e^x .

Note: if we don't include the lines $y=x$ and $y=x+2\pi$, then the image should not include the curve $e^x \cdot e^{ix}$ ($x \in \mathbb{R}$) in the w -plane. — a spiral.

3. Pf $f(z) = \frac{a_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$ ($0 < |z-z_0| < R$)

where $a_1 \neq 0$. So, $f(z) = \frac{1}{z-z_0} g(z)$ where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Thus, for z near z_0 :

$$f'(z) = -\frac{1}{(z-z_0)^2} g(z) + \frac{1}{z-z_0} g'(z)$$

$$= \frac{1}{(z-z_0)^2} \underbrace{[(z-z_0)g'(z) - g(z_0)]}_{\text{Let this be } h(z)}.$$

$$h(z_0) = -g(z_0) \neq 0.$$

(3)

Hence $\lim_{z \rightarrow z_0} |f'(z)| = \infty$. Or $|f'(z)| \geq \frac{c}{|z-z_0|^2}$

This means that $f' \neq 0$ near z . Hence it is 1-1 in a punctured disk at z_0 .

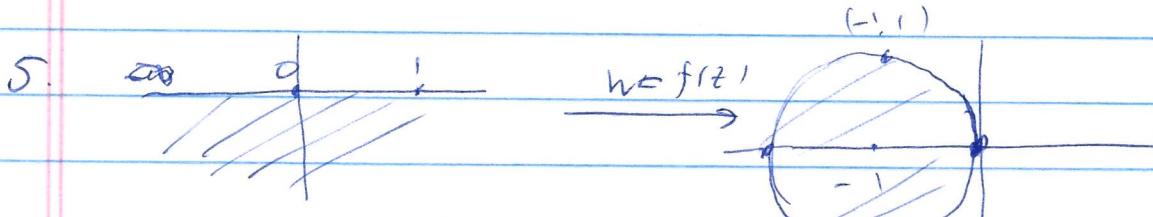
for some $c > 0$ and
~~for $0 < |z-z_0| < r$~~
 for some $r > 0$.

4. $w = az + b$.

$$z=0 \Leftrightarrow w=s \Rightarrow s=b \Rightarrow w = az+s$$

$$z=i \Leftrightarrow w=2 \Rightarrow 2 = ai + s, a = 3i$$

$$w = 3i z + s.$$



$$z_1=1, z_2=0, z_3=\infty$$

$$(-1, 1)$$

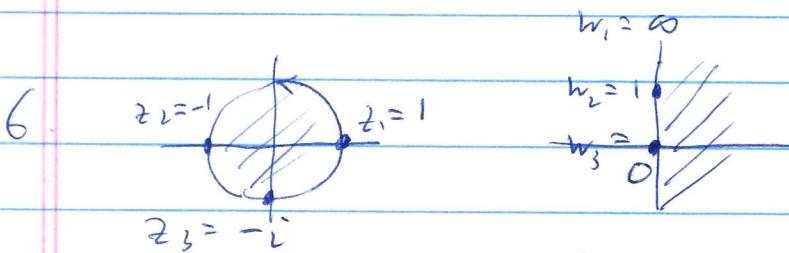
$$w_1=0, w_2=-1+i, w_3=-2$$

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w(-1+i+2)}{(w+2)(-1+i)} = \frac{z-1}{0-1} = 1-z$$

$$\frac{w(1+i)}{(w+2)(-1+i)} = 1-z \quad w = \frac{-2z+2}{z-1-i}$$



$$z_3=-i$$

$$\frac{w_2-w_3}{w-w_3} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{1}{w} = \frac{(z-1)(-1+i)}{(z+i)(-1-i)} = \frac{(z-1)(i-1)}{(z+i)(-2)}$$

$$w = \frac{(1+i)(z+i)}{z-1}$$

[4]

7. $z_1 = 0, z_2 = \lambda, z_3 = \infty$

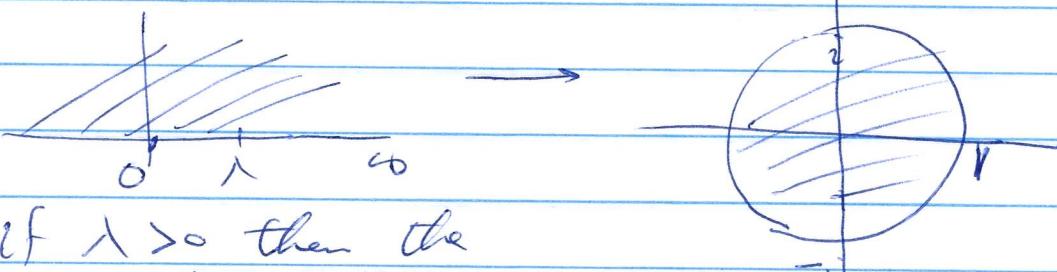
$w_1 = -i, w_2 = 1, w_3 = i$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+i)(1-i)}{(w-i)(1+i)} = \frac{z}{\lambda}$$

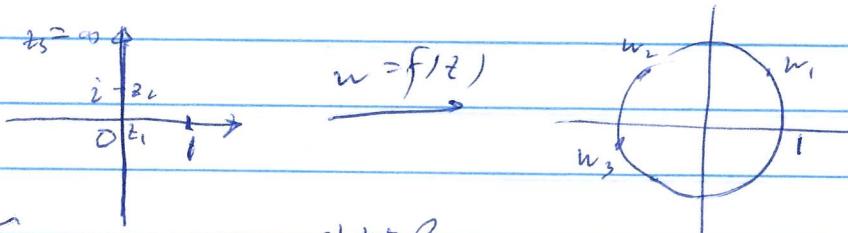
$$\frac{w+i}{w-i} \frac{(-i)}{(1-i)} = \pm \frac{z}{\lambda}, \quad \frac{w+i}{w-i} = \frac{-iz}{\lambda}$$

$$w = \frac{iz + \lambda}{z + i\lambda}$$



If $\lambda > 0$ then the upper-half plane is mapped to $|w| < 1$.
 $[\lambda < 0 \Rightarrow |w| > 1]$.

8.



$$[w = f(z) = \frac{\alpha z + \beta}{z - 1} \text{ since } f(1) = \infty]$$

choose $z_1 = 0, z_2 = i, z_3 = \infty$.

$$w_1 = 1, w_2 = i, w_3 = -1$$

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{z}{i} = \frac{w-1}{w+1} \frac{i+1}{i-1} \quad z = \frac{w-1}{w+1}$$

$$w = \frac{z+1}{z-1}. \quad w(z = -1) = 0.$$