

Solns to HW # 7 Problems

1. (1) $\frac{dx}{dt} = -\frac{1}{2x} \quad -2x dx = dt$

$-x^2 = t + C \quad -1 = -x(0)^2 = 0 + C \quad C = -1$
 $x^2 = -t + 1 \quad x = \pm \sqrt{1-t}$

Since $x(0) = 1$, we choose "+". So, the solution is $x(t) = \sqrt{1-t}$.
 max. interval: $(-\infty, 1)$.
 $\lim_{t \rightarrow -\infty} x(t) = +\infty$
 $\lim_{t \rightarrow 1^-} x(t) = 0$

(2) $\frac{dx}{dt} = \frac{1}{x^2} \quad x^2 dx = dt, \quad \frac{1}{3} x^3 = t + C$
 $\frac{1}{3} \cdot 1^3 = \frac{1}{3} [x(1/3)]^3 = \frac{1}{3} + C \quad C = 0$
 $x^3 = 3t \quad x = (3t)^{1/3}$

Note that the soln $x = x(t)$ should be differentiable. So, the interval should be $(-\infty, 0)$ or $(0, \infty)$ [only one of them]. But the initial condition is $x(1/3) = 1$. $1/3 \in (0, \infty)$. Hence, the max. interval of solution $x(t) = (3t)^{1/3}$ is $(0, \infty)$.
 $\lim_{t \rightarrow 0^+} x(t) = 0, \quad \lim_{t \rightarrow \infty} x(t) = +\infty$

(3) $x_3 = 1 \Rightarrow x_3 = t + c_3. \quad x_3(1/\pi) = 1/\pi \Rightarrow c_3 = 0$
 So, $x_3(t) = t$

$$\begin{cases} \dot{x}_1 = -x_2/t^2 & x_1(\sqrt{\pi}) = 0 \\ \dot{x}_2 = x_1/t^2 & x_2(\sqrt{\pi}) = -1. \end{cases}$$

Let $\tau = 1/t$ ($t > 0$). $y_1(\tau) = x_1(t)$, $y_2(\tau) = x_2(t)$.

$$\frac{dx_1}{dt} = \frac{dy_1}{d\tau} \frac{d\tau}{dt} = -\frac{1}{t^2} \frac{dy_1}{d\tau}, \quad \frac{dx_2}{dt} = -\frac{1}{t^2} \frac{dy_2}{d\tau}$$

$$\begin{cases} \frac{dy_1}{d\tau} = y_2 & y_1(\pi) = 0 \\ \frac{dy_2}{d\tau} = -y_1 & y_2(\pi) = -1. \end{cases}$$

~~$y_1'' - y_2' = -y_1$~~

$$\frac{d}{d\tau} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\lambda_1, \lambda_2 = \pm i. \quad \bar{u}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \bar{u}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{aligned} e^{i\tau} \begin{bmatrix} 1 \\ i \end{bmatrix} &= (\cos \tau + i \sin \tau) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \cos \tau \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin \tau \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \left(\sin \tau \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos \tau \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} \cos \tau \\ -\sin \tau \end{bmatrix} + i \begin{bmatrix} \sin \tau \\ \cos \tau \end{bmatrix} \end{aligned}$$

$$\text{So, } \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} \cos \tau \\ -\sin \tau \end{bmatrix} + c_2 \begin{bmatrix} \sin \tau \\ \cos \tau \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 1 \end{matrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y_1(\tau) \\ y_2(\tau) \end{bmatrix} = \begin{bmatrix} \sin \tau \\ \cos \tau \end{bmatrix} = \begin{bmatrix} \sin \frac{1}{t} \\ \cos \frac{1}{t} \end{bmatrix}$$

Solu: $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \sin 1/t \\ \cos 1/t \end{bmatrix}$ max. interval $(0, \infty)$

$\lim_{t \rightarrow 0} x(t)$: does not exist.

$\lim_{t \rightarrow \infty} x(t)$: does not exist.

only: $x_2(\infty) = 0$

Note: $x_1 \dot{x}_1 + x_2 \dot{x}_2 = 0 \Rightarrow (x_1^2 + x_2^2)' = 0 \quad x_1^2 + x_2^2 = C$

Initial condition: $\Rightarrow C = 1$. So, $x_1(t) = \cos f(t)$ or $x_1(t) = \sin f(t)$
 still, cannot find $f(t)$. $x_2(t) = \sin f(t)$ or $x_2(t) = \cos f(t)$.

2. $\dot{x}_1 = -x_1 \Rightarrow x_1(t) = c_1 e^{-t} \quad c_1 = x_1(0)$

$\dot{x}_2 = x_1^2 + 2x_2 \Rightarrow \dot{x}_2 - 2x_2 = c_1^2 e^{-2t}$

$e^{-2t}(\dot{x}_2 - 2x_2) = c_1^2 e^{-4t}$

$(e^{-2t} x_2)' = c_1^2 e^{-4t}$

$e^{-2t} x_2(t) - e^{-2 \cdot 0} x_2(0) = c_1^2 \left(-\frac{1}{4}\right) (e^{-4t} - 1)$

$x_2(t) = -\frac{1}{4} c_1^2 (e^{-2t} - e^{2t}) + c_2 e^{2t} \quad c_2 = x_2(0)$

If $(c_1, c_2) \in S$, i.e., $c_2 = -c_1^2/4$. then

$x_2(t) + [x_1(t)]^2/4$

$= -\frac{1}{4} c_1^2 (e^{-2t} - e^{2t}) + c_2 e^{2t} + \frac{c_1^2}{4} e^{-2t} = 0$

Hence, $x_2(t) = -x_1(t)^2/4$, i.e., $(x_1(t), x_2(t)) \in S$.

Hence, S is invariant under the flow $\phi_t(x)$.

3. (1) $\begin{cases} \dot{x}_1 = x_1 - x_1 x_2 \\ \dot{x}_2 = x_2 - x_1^2 \end{cases} \quad \begin{matrix} x_1 - x_1 x_2 = 0 & x_1(1-x_2) = 0 \\ x_2 - x_1^2 = 0 & x_2 = x_1^2 \end{matrix}$

$x_1 = 0, x_2 = 0. \quad (0, 0)$

3 eq. pts.

$x_2 = 1, x_1 = \pm 1. \quad (1, 1), (-1, 1)$

$(0, 0) \quad A = \begin{bmatrix} 1-x_2 & -x_1 \\ -2x_1 & 1 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ source linearly unstable

$(1, 1) \quad A = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix} \quad \lambda^2 - \lambda - 2 = 0. \quad \lambda_1 = 2, \lambda_2 = -1.$
saddle. linearly unstable.

$(-1, 1) \quad A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ same: $\lambda_1 = 2, \lambda_2 = -1.$
saddle

(2) $\begin{cases} \dot{x}_1 = -4x_2 + 2x_1 x_2 - 8 \\ \dot{x}_2 = 4x_2^2 - x_1^2 \end{cases}$

$x_1 = \pm 2x_2$

$x_1 = 2x_2 \quad -4x_2 + 2 \cdot 2x_2 \cdot x_2 - 8 = 0 \quad x_2^2 - x_2 - 2 = 0$

$x_2 = 2, -1.$

$(4, 2), (-2, -1)$

$x_1 = 4, -2$

$$x_1 = -2x_2 \quad -4x_2 - 4x_2^2 - 8 = 0 \quad x_2^2 + x_2 + 2 = 0$$

no real roots. So, two crt pts $(4, 2), (-2, -1)$.

$$(4, 2) \quad A = \begin{bmatrix} 2x_2 & -4+2x_1 \\ -2x_1 & 8x_2 \end{bmatrix}_{(4,2)} = \begin{bmatrix} 4 & 4 \\ -8 & 16 \end{bmatrix}$$

$$\lambda^2 - 20\lambda + 96 = 0. \quad \lambda_{1,2} = \frac{20 \pm \sqrt{200 - 4 \times 96}}{2}$$

unstable spirals.

$$= 10 \pm i\sqrt{96-50}$$

$$= 10 \pm i\sqrt{46}$$

$$(-2, -1) \quad A = \begin{bmatrix} -2 & 0 \\ 4 & -8 \end{bmatrix} \quad \lambda_1 = -2, \lambda_2 = -8$$

stable node.

The only crt pt is $(0, 0)$.

$$4. (1) \quad \dot{x}_1, \dot{x}_2 = -x_1^2 + x_1x_2 + x_1^2x_2$$

$$x_2\dot{x}_2 = x_1x_2 - x_2^2 - x_1^2x_2 - x_2^4$$

$$\text{So, } (x_1^2 + x_2^2)' = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2(-x_1^2 - x_2^2 - x_2^4) < 0$$

$V(x_1, x_2) = x_1^2 + x_2^2$ is a Liapunov function. if $x \neq 0$

So, $(0, 0)$ is asymptotically stable

Why only one crt pt? Check it.

$$-x_1 + x_2 + x_1x_2 = 0. \quad x_1(x_2 - 1) + x_2 = 0.$$

Clearly if $x_2 \neq 1$. So, $x_1 = \frac{-x_2}{x_2 - 1} = \frac{x_2}{1 - x_2}$.

Plug this into the r.h.s. of 2nd eq.

$$\frac{x_2}{1-x_2} - x_2 - \left(\frac{x_2}{1-x_2}\right)^2 - x_2^4 = 0$$

$$x_2 = 0. \quad \text{+ 1st eq. } \Rightarrow x_1 = 0. \quad (0, 0).$$

$$x_2 \neq 0. \quad \frac{1}{1-x_2} - 1 - \frac{x_2}{(1-x_2)^2} - x_2^4 = 0$$

$$(1-x_2) - (1-x_2)^2 - x_2 - (1-x_2)^2 x_2^4 = 0$$

$$-2 + 2x_2 - x_2^2 = 0 \quad \text{no real roots!}$$

(2) Find all the crt pts.

$$\begin{cases} -x_1 - 2x_2 + x_1x_2^2 = 0 & x_1(x_2^2 - 1) = 2x_2 \\ 3x_1 - 3x_2 + x_2^3 = 0 \end{cases}$$

clearly $x_2^2 \neq 1$. So, $x_1 = \frac{2x_2}{x_2^2 - 1}$

$$\frac{6x_2}{x_2^2 - 1} - 3x_2 + x_2^3 = 0$$

$x_2 = 0 \Rightarrow x_1 = 0$. $(0,0)$ is a crt pt.

$$x_2 \neq 0. \quad \frac{6}{x_2^2 - 1} - 3 + x_2^2 = 0, \quad 6 - 3(x_2^2 - 1) + x_2^2(x_2^2 - 1) = 0$$

$$x_2^4 - 4x_2^2 + 9 = 0 \quad (-4)^2 - 4 \cdot 9 < 0 \quad \text{no real roots.}$$

$$(0,0) \quad A = \begin{bmatrix} -1 + x_2^2 & -2 + 2x_1x_2 \\ 3 & -3 + 3x_2^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}$$

$$\lambda^2 + 4\lambda + 9 = 0. \quad \lambda_{1,2} = \frac{-4 \pm \sqrt{-20}}{2} = -2 \pm i\sqrt{5}$$

(linearly stable spiral.

$$3x_1\dot{x}_1 = -3x_1^2 - 6x_1x_2 + 3x_1^2x_2^2$$

$$2x_2\dot{x}_2 = 6x_1x_2 - 6x_2^2 + 2x_2^4$$

$$\text{Let } V(x_1, x_2) = 3x_1^2 + 2x_2^2.$$

Then, in $E =$ a small disk centered at $(0,0)$

$$= \{(x_1, x_2) : x_1^2 + x_2^2 < \delta^2\}$$

where $\delta > 0$ is small (to be determined).

We have $V > 0$ in E except $V = 0$ at $(0,0)$

V is smooth.

$$\dot{V} = 6x_1\dot{x}_1 + 4x_2\dot{x}_2 = 2(3x_1\dot{x}_1 + 2x_2\dot{x}_2)$$

$$= 2(-3x_1^2 - 6x_2^2 + 3x_1^2x_2^2 + 2x_2^4)$$

$$\leq -6x_1^2 - 12x_2^2 + 6x_1^2x_2^2 + 6x_2^4$$

$$\text{If } (x_1, x_2) \in E, \text{ i.e., } \Rightarrow \leq -6x_1^2 - 12x_2^2 + 6x_2^2(x_1^2 + x_2^2)$$

$$\leq -6x_1^2 - 12x_2^2 + 6x_2^2 \cdot \delta^2$$

$$\leq -6x_1^2 - 6x_2^2 \quad \text{if } \delta \in (0,1).$$

$$< 0 \quad \text{if } (x_1, x_2) \neq (0,0). \quad (x_1, x_2) \in E.$$

(6)

Therefore $V(x_1, x_2)$ is a strict Liapunov function for $(0,0)$. So, $(0,0)$ is asymptotically stable.

5. Proof. Since \vec{x}_0 is a strictly local min of V , $\nabla V(\vec{x}_0) = \vec{0}$ so, \vec{x}_0 is a crit. pt of $\dot{\vec{x}} = -\nabla V(\vec{x})$. Let $W(\vec{x}) = V(\vec{x}) - V(\vec{x}_0)$, $x \in B(\vec{x}_0, \delta) = E$ where $\delta > 0$ is such that $W(\vec{x}) > 0 \quad \forall \vec{x} \in E, \vec{x} \neq \vec{x}_0$ and $\nabla W(\vec{x}) \neq \vec{0} \quad \forall \vec{x} \in E, \vec{x} \neq \vec{x}_0$. This is possible since \vec{x}_0 is a strictly local min of $V(\vec{x})$.

Thus $W: E \rightarrow \mathbb{R}$ is smooth. $W > 0 \quad \forall \vec{x} \in E \setminus \{\vec{x}_0\}$

$$\dot{W}(\vec{x}) = \nabla V(\vec{x}) \cdot (-\nabla V(\vec{x})) = -|\nabla V(\vec{x})|^2 < 0$$

$\forall \vec{x} \in E \setminus \{\vec{x}_0\}$, strict

Hence, W is a Liapunov function for \vec{x}_0 .

6. Proof $V(x_1, x_2) = x_1^2(x_1-1)^2 + x_2^2$ $x_1((x_1-1)^2 + x_1(x_1-1)) = x_1(x_1-1)(2x_1-1)$

$$\nabla V(x_1, x_2) = \begin{bmatrix} 2x_1(x_1-1)^2 + 2x_1^2(x_1-1) \\ 2x_2 \end{bmatrix}$$

$$\nabla V = \vec{0} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

So, $(0,0), (1,0), (1/2,0)$ are the crit. pts of V .

The Hessian $\nabla^2 V = \begin{bmatrix} 12x_1^2 - 12x_1 + 2 & 0 \\ 0 & 2 \end{bmatrix}$

$$\nabla^2 V(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ sym. positive definite.}$$

$$\nabla^2 V(1,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ sym. positive def.}$$

So, both $(0,0), (1,0)$ are strictly local min of V .

$$\nabla^2 V(1/2, 0) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad \lambda_1 < 0, \lambda_2 > 0$$

So, a saddle pt of V .

By the result of Prob. #5, $(0,0), (1,0)$ are asymptotically stable.

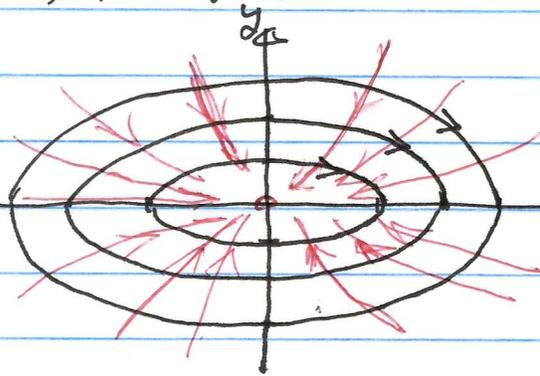
At $(1/2, 0)$

$$\vec{x}' = -\nabla V(\vec{x}) = -x_1^2 + 2x_2^2 + \text{higher order terms.}$$

Hence, $(1/2, 0)$ is a saddle pt.

7 (1) $H(x,y) = x^2 + 2y^2$

$H(x,y) = C$.
larger C
larger curves



Each curve is: $H(x,y) = \text{const.}$
 $x = x(t), y = y(t)$ of the Hamiltonian system satisfy

$$H(x(t), y(t)) = \text{const.}$$

[Energy conservation]

Why clockwise?

$$\begin{cases} \dot{x} = 4y \\ \dot{y} = -2x \end{cases} \text{ at } (1,0). \begin{cases} \dot{x} = 0 \\ \dot{y} = -2. \end{cases}$$

Red curves are trajectories of the gradient system $\begin{cases} \dot{x} = -\partial_x H \\ \dot{y} = -\partial_y H \end{cases}$ orthogonal to $\{H = \text{const.}\}$. from high H -values to low H -values.

(2)

The Hamiltonian system

$$\begin{cases} \dot{x} = -2y \\ \dot{y} = -2x \end{cases}$$

The gradient system

$$\begin{cases} \dot{x} = -2x \\ \dot{y} = 2y \end{cases}$$

