Math 210: Mathematical Methods for Physical Sciences and Engineering

Lecture Notes on Complex Analysis

Plan. 3 - 4 Weeks

1. Analytic Functions
   (a) Limits and continuity
   (b) Analyticity. The Cauchy-Riemann Equations
   (c) Elementary Analytic Functions (trig., exp., log. functions)

2. Complex Integration
   (a) Definition of Contour Integrals.
   (b) Cauchy's Integral Theorem / Formula.

3. Series Representation (for Analytic Functions)
   (a) Taylor Series / Power Series
   (b) Laurent Series
   (c) Zeros and Singularities

4. Residue Theory
   (a) The Residue Theorem
   (b) Techniques of Integration

5. Conformal Mapping (If time permits)
   (a) Concept
   (b) Möbius Transformations
   (c) The Schwarz-Christoffel Transformation
   (d) Applications
Section 1. Analytic Functions

Section 1.1. Complex-Valued Functions, Limits, and Continuity

Notation: \( \mathbb{R} = \{ \text{all real numbers} \} \)
\( \mathbb{C} = \{ \text{all complex numbers} \} \)

Complex number \( z = a + bi \), \( a, b \in \mathbb{R} \)
- real part: \( \text{Re} \ z = a \)
- imaginary part: \( \text{Im} \ z = b \)
- \( i^2 = -1 \)
- absolute value or modulus: \( |z| = \sqrt{a^2 + b^2} \)
- the complex conjugate: \( \bar{z} = a - bi \)
- \( |\bar{z}| = |z| \)
- \( \bar{z}_1 \cdot \bar{z}_2 = \bar{z}_1 \cdot \bar{z}_2 \)
- \( \frac{1}{\bar{z}_2} = \bar{z}_2 |\bar{z}_2|^2 \)
- \( |z_1 \cdot z_2| = |z_1| |z_2| \)
- \( |\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|} \)

\[ z = r e^{i \phi} \]
\[ r = |z| = \sqrt{a^2 + b^2} \]
\[ \phi = \text{arg} \ z = \text{arctan} \frac{b}{a} \]
\[ \text{arg} i = \frac{\pi}{2} + 2\pi k \quad (k = 0, \pm 1, \pm 2, \ldots) \]
\[ \text{arg}(z_1 z_2) = \text{arg} z_1 + \text{arg} z_2 \]

- \( e^z = e^{a+bi} = e^a \cdot e^{bi} = e^a \left( \cos b + i \sin b \right) \)

If \( m \) is an integer, then:
\[ y_m = (e^{i \cdot \frac{2\pi k}{m}}) y_m = e^{i \frac{2\pi k}{m}} = e^{2\pi k i m} + i \sin \frac{2\pi k}{m} \quad (k = 0, 1, 2, \ldots, m-1) \]
Complex function (or: Complex-valued function).

\[ w = f(z) \]

\[ z = x + iy \]

\[ w = u(x, y) + iv(x, y) \]

Examples:

1. \[ w = z^2 + 2z = (x + iy)^2 + 2(x + iy) \]
   \[ = x^2 - y^2 + 2xyi + 2x + 2yi \]
   \[ = (x^2 - y^2 + 2x) + (2xy + 2y)i \]
   \[ = u(x, y) + iv(x, y) \]

2. \[ f(z) = z^2 + 2i \]
   \[ = \frac{z + \bar{z}}{2} + 2i \]
   \[ \begin{cases} u(x, y) = x^2 \\ v(x, y) = 2 \end{cases} \]

If \( |z| \leq 1 \) then \( 0 \leq u(x, y) \leq x^2 \).

The unit disk \( |z| \leq 1 \) is mapped by \( f(z) = z^2 + 2i \) to the line segment \([2i, 1 + 2i]\).

3. \( f(z) = z^3 \).
Limits

**Theorem** \( \lim_{n \to \infty} z_n = x_n + iy_n \quad (n = 1, 2, \ldots) \)

\[
\lim_{n \to \infty} z_n = z = x + iy \iff \lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y.
\]

**Definition** For any \( \varepsilon > 0 \), there exists \( N \) (integer), such that \( n \geq N \iff |z_n - z| < \varepsilon \).

Let \( w = f(z) \) defined on \( 0 < |z - z_0| < \delta \).

**Definition** \( \lim_{z \to z_0} f(z) = w_0 \)

if for any \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) \) such that \( 0 < |z - z_0| < \delta \implies |f(z) - w_0| < \varepsilon \).

**Theorem** \( \lim_{z \to z_0} f(z) = w_0 \quad f(z) = u(x, y) + iv(x, y) \)

\[
w_0 = u_0 + iv_0 \quad z_0 = x_0 + iy_0.
\]

**Definition** If \( \lim_{z \to z_0} f(z) = f(z_0) \), then \( f(z) \) is continuous at \( z_0 \).

**Theorem** \( f(z) = u(x, y) + iv(x, y) \) is continuous at \((x_0, y_0)\) \iff \( u, v \) are continuous at \((x_0, y_0)\).

**Theorem** If \( \lim_{z \to z_0} f(z) = \alpha \), \( \lim_{z \to z_0} g(z) = \beta \), then

\[
\lim_{z \to z_0} \left[ f(z) \pm g(z) \right] = \alpha \pm \beta,
\]

\[
\lim_{z \to z_0} \left[ f(z) g(z) \right] = \alpha \beta \quad \text{and} \quad \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{\alpha}{\beta} \quad \text{(if} \beta \neq 0)\]
Definition. Let \( f(z) \) be defined in a neighborhood of \( z_0 \) (i.e., \( 1 > |z - z_0| > \delta_0 \) for some \( \delta_0 > 0 \)). The derivative of \( f \) at \( z_0 \) is
\[
 f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},
\]
if the limit exists. In this case, we say that \( f \) is differentiable at \( z_0 \).

Examples:
1. \( f(z) = \bar{z}, \quad |z|, \quad \text{Re} z, \quad \text{Im} z \) are all continuous but nowhere differentiable!
   Consider \( f(z) = \bar{z} \).
\[
 f(z_0 + \Delta z) - f(z_0) = \frac{\Delta \bar{z}}{\Delta z} \Delta z = \Delta \bar{z}.
\]
   If \( \Delta z = \Delta x + i \Delta y \in \mathbb{C} \), then \( \Delta \bar{z} = \Delta z \cdot \frac{\Delta \bar{z}}{\Delta z} = 1 \).
   If \( \Delta z = i \Delta y \in \mathbb{C} \), then \( \Delta \bar{z} = -\Delta z \cdot \frac{\Delta \bar{z}}{\Delta z} = -1 \).
   So, the limit as \( \Delta z \to 0 \) does not exist.

2. \((z^n)' = n z^{n-1}\).

\[
 f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0,
\]
\[
 f'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \ldots + a_1,
\]
\[
\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{\Delta z} \left[ \begin{array}{c}
 (n) z^{n-1} \Delta z + O((\Delta z)^2) \\
 + \ldots + (n-1) \Delta z^{n-2} + (\Delta z)^n - z^n
\end{array} \right]
\]
\[
= \frac{1}{\Delta z} \left[ (n) z^{n-1} \Delta z + O((\Delta z)^2) \right] = n z^{n-1} + O(\Delta z), \quad \text{as} \quad \Delta z \to 0.
\]

Operations: \((f \pm g)'(z) = f'(z) \pm g'(z)\), \((fg)' = f'g + fg'\), \((f/g)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}\).

Chain rule: \((f(g(z)))' = f'(g(z))g'(z)\).
The Riemann equations

Let \( f(z) = u(x, y) + i v(x, y) \) be differentiable at \( z = x + iy \). Then

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{(The C.-R. equations)}
\]

Moreover, \( f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} - i \frac{\partial v}{\partial y} \).

Example (1) Check if \( f(z) = \text{Re} z = x \) is differentiable or not.

\( u(x, y) = x, \ v(x, y) = 0. \)
\[
\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 0. \quad \text{No!}
\]

(2) Do the same for \( f(z) = (x^2 + y) + i(y^2 - x). \)

\( u(x, y) = x^2 + y, \ v(x, y) = y^2 - x \)
\[
\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2y. \quad \text{C.-R. equations true for } x = y.
\]

Proof By definition, \( f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \) exists.

\( \Delta z = \Delta x + i \Delta y. \) Choose \( \Delta y = 0. \) \( \Delta z = \Delta x \in \mathbb{R}. \)

\[
f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) + i v(x + \Delta x, y) - u(x, y) - i v(x, y)}{\Delta x}
= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}
= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.
\]

Choose \( \Delta z = 2\Delta y \) (i.e., \( \Delta x = 0 \)). Then
\[ f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i \Delta y} \]
\[ = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}. \]
Hence, \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \), \( \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \)
and \( f'(z) = u_x + i u_y = v_y - i v_x \]
\[ [= u_x - i u_y = v_y + i v_x]. \] \[ \text{O.F.D.} \]

**Definition** A subset \( G \) of \( \mathbb{C} \) is open, if for any \( z \in G \), there exists a \( \delta = \delta(z) > 0 \) s.t. the disk \( \{ \bar{z} : |z - z_0| < \delta \} \subseteq G \). i.e., if \( G \) contains a point, then \( G \) contains a disk centered at that point.

**Definition** \( w = f(z) \) is analytic in an open set \( G \subseteq \mathbb{C} \), if it is differentiable at every point in \( G \).

**Example** \( f(z) = x^2 + y + i (y^2 - x) \)
\( u_x = 2x, \quad v_y = 2y, \quad u_y = 1, \quad v_x = -1. \)
C. - R. equations hold for \( x = y \), not open.
So, \( f \) is not analytic in any open set.
Theorem. Let \( f(z) = u(x, y) + iv(x, y) \) be defined in an open set \( G \), and \( z_0 \in G \). Suppose

1. \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \) exist in \( G \), and are continuous at \( (x_0, y_0) \);

2. The Cauchy–Riemann equations hold true for \( f \) at \( z_0 \).

Then, \( f \) is differentiable at \( z_0 \).

In particular, if all \( u_x, u_y, v_x, v_y \) exist and are continuous in \( G \), then \( f(z) \) is analytic in \( G \).

Proof.

\[
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{[u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y}
\]

\( \Delta z \)-part:

\[
u - part: u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) \]

\[
= \frac{[u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)]}{\Delta x} + \frac{[u(x_0, y_0 + \Delta y) - u(x_0, y_0)]}{\Delta y} \]

\[
= \Delta x \left[ u_x (x_0, y_0) + O(1) \right] + \Delta y \left[ u_y + \frac{i}{\Delta y} \left( \Delta x \right) + O(1) \right] \]

\[
= \Delta x \left[ u_x + \frac{i}{\Delta y} \left( \Delta x \right) + O(1) \right] + i\Delta y \left[ v_x + \frac{i}{\Delta y} \left( \Delta x \right) + O(1) \right] \]

\[
\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x [u_x + \frac{i}{\Delta y} \left( \Delta x \right) + O(1)] + i\Delta y [v_x + \frac{i}{\Delta y} \left( \Delta x \right) + O(1)]}{\Delta x + i\Delta y} \]

\[= \frac{\Delta x [u_x + i\Delta y + O(1)]}{\Delta x + i\Delta y} \]

\[= u_x + i\Delta y + O(1) \quad \Delta z \to 0.
\]
Definition: \( w = f(z) \) is an **entire function** if it is analytic in \( \mathbb{C} \).

**Example**

\[
\begin{align*}
f(z) &= e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) \\
u(x, y) &= e^x \cos y \\
v(x, y) &= e^x \sin y. \\
u_x &= e^x \cos y. \\
u_y &= e^x \sin y. \\
u_x = v_y \\
v_y &= -e^x \cos y. \\
v_x = e^x \sin y. \\
v_y = -v_x.
\end{align*}
\]

So, \( f(z) = e^z \) is an entire function.

**Theorem**

If \( f(z) = u(x, y) + iv(x, y) \) is analytic in an open set \( G \), then both \( u \) and \( v \) are harmonic in \( G \).

**Proof**

We will show later that in this case \( u, v \) are infinitely differentiable. So,

\[
\begin{align*}
u_x &= \nu_y \\
v_{xx} &= \nu_{yy} = -\nu_{yy}.
\end{align*}
\]

i.e., \( \Delta u = u_{xx} + u_{yy} = 0 \).

Same for \( \Delta v = 0 \). \( \Box \).

**Application**

D: open, connected. \( f \) is analytic in \( D \).

\( f(z) = 0 \Rightarrow f = \text{const.} \)
Section 1.3 Elementary Functions

Exponential function

\[ w = e^z = e^{x + iy} = e^x (\cos y + i \sin y) \]

- \[ |e^z| = e^x \]
- \[ \arg e^z = y + 2k\pi \quad (k \in \mathbb{Z}) \]
- \[ Z = \{ \text{all integers} \} = \{ 0, \pm 1, \pm 2, \ldots \} \]
- \[ e^{z_1} = e^{z_2} \iff z_1 - z_2 = 2k\pi i \quad (k \in \mathbb{Z}) \]

Hence, \( e^z \) is \( 2\pi i \)-periodic.

- \( (e^z)' = e^z \). \( e^z \) is an entire function.
Trigonometric Functions

Definition: \[ \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \]

\[ \tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z} \]

\[ \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z} \]

1. \((\sin z)') = \cos z, \quad (\cos z)' = -\sin z\]
2. \((\tan z)' = \sec^2 z, \quad (\cot z)' = -\csc^2 z\]
3. \[ \sin (z + 2\pi) = \sin z, \quad \cos (z + 2\pi) = \cos z \]
   \[ \sin^2 z + \cos^2 z = 1 \]

\[ \sin (-z) = -\sin z, \quad \cos (-z) = \cos z \]
\[ \sin 2z = \cos^2 z - \sin^2 z, \quad \cos 2z = 2\cos^2 z - 1, \quad \sin 2z = 2\sin z \cos z \]
\[ \cos 2z = \cos^2 z + \sin^2 z, \quad \sin (z + 2\pi) = \sin z, \quad \cos (z + 2\pi) = \cos z \]

Hyperbolic Functions

\[ \sinh z = \frac{e^z - e^{-z}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2} \]

1. \((\sinh z)') = \cosh z, \quad (\cosh z)' = \sinh z\]
2. \[ \sinh z = -i \sin i z, \quad \cosh z = \cos i z \]
The Logarithmic Function

Try: \( w = \log z \) if \( z = e^w \)

Only for \( z \neq 0 \) (since \( e^w \neq 0 \) for \( w \))

Write \( z = r e^{i\theta} \), \( w = u + i\nu \). Then \( z = e^w \Rightarrow \)

\( re^{i\theta} = e^u e^{i\nu} \)

\( \Rightarrow r = e^u \), i.e. \( u = \log r \)

\( e^{i\alpha} = e^{i\nu} \)

\( \nu = \alpha = \arg z \) **multivalued**

**Definition** If \( z \neq 0 \) then

\[ \log z = \log |z| + i \arg z \]

\[ = \log |z| + i \operatorname{Arg} z + i 2k\pi \]

\( (k = 0, \pm 1, \pm 2, \ldots) \)

[ \( \operatorname{Arg} z \): called the principal value of argument, is defined to be the argument such that \( -\pi < \operatorname{Arg} z \leq \pi \). ]

**Define:** \( \log z = \log |z| + i \operatorname{Arg} z \)

Call it the principal value of the logarithm \( \log z \).

**Theorem** Let \( D^* \) be the domain consisting all \( z \in \mathbb{C} \) except those on the negative real axis. Then \( \log z \) is analytic in \( D^* \) and

\[ \frac{\partial}{\partial \bar{z}} \log z = \frac{1}{z}, \quad \text{if } z \in D^* \]
Corollary
1. \( \text{Arg} z \) is harmonic in \( D^* \).
2. \( \log |z| \) is harmonic in \( C \setminus \{0\} \).

Example
\[
\begin{align*}
  f(z) &= \log(3z+i) \\
  f'(z) &= \frac{1}{3z+i} \cdot (3z+i) \\
  &= \frac{3}{3z+i}.
\end{align*}
\]

A branch of \( \log z \): Let \( \tau \in \mathbb{R} \).
\[
L_{\tau}(z) = \log |z| + i \text{arg} z \\
  \text{if } \text{arg} z \in (\tau, \tau + 2\pi].
\]
This is a single-valued continuous function.
\[
\frac{df}{dz} L_{\tau}(z) = \frac{1}{z}.
\]

Definition
\( F(z) \) is a branch of a multi-valued function \( f(z) \) in a domain \( D \), if \( F(z) \) is a single-valued and continuous in \( D \), and has the property that, for any \( z \in D \), the value of \( F(z) \) is one of the values of \( f(z) \).

Example
\begin{align*}
\text{Arg} z & \text{ is a branch of arg} \ z \text{ in } D^* \\
\log z & \text{ is a branch of log} z \text{ in } D^* \\
\exp(\frac{1}{2}z) & \log z & \text{ is a branch of } z^{\frac{1}{2}} \\
& \text{ in the right half-plane.}
\end{align*}
Example. Determine a branch of \( f(z) = \log(z^3 - 2) \) that is analytic at \( z = 0 \) and find \( f(0) \), \( f'(0) \).

Solution. By the chain rule, it suffices to choose any branch of logarithm that is analytic at \( g(0) = -2 \) where \( g(z) = z^3 - 2 \). In particular,
\[
\tilde{f}(z) = \frac{z - \frac{2}{3}}{\sqrt[3]{-2}}
\]
should work.
\[
\tilde{f}(0) = \frac{0 - \frac{2}{3}}{\sqrt[3]{-2}} = \frac{2}{3} \log 2 + i\pi
\]
\[
\tilde{f}'(0) = \frac{g'(0)}{g(0)} = \frac{g'(0)}{g(0)} = 0
\]

Complex Powers Functions

Definition. If \( x, y \in \mathbb{C}, y \neq 0 \), then
\[
\begin{align*}
    x^y &= e^{y \log x}.
\end{align*}
\]

Example. 1. \( (-2)^i = e^{i \log(-2)} = e^{i \log 2 - \pi - 2\pi k} \) as \( \log(-2) = \log 2 + (1 + 2\pi k)i \), \( k \in \mathbb{Z} \)

2. \( w^n = e^{\frac{m}{n} \log |w|} \exp \left( i \frac{m}{n} (\arg w + 2\pi n) \right) \)

Note: 1. If \( x \in \mathbb{R} \) (real), \( x^z \) is single-valued.
2. If \( x = \frac{m}{n} \), \( m, n \in \mathbb{Z}, n \neq 0 \), then \( x^z \) is finitely many values.
3. \( x^z \) is infinitely many values for otherwise.
Example: Define a branch of \((z^2-1)^{1/2}\) that is analytic in the exterior of the unit disc \(|z|<1\).

Solution: Need to find \(w = f(z)\) analytic outside unit circle and

\[ w = z^2 - 1. \]

The principal branch of \((z^2-1)^{1/2}\) is

\[ e^{(1/2) \log (z^2-1)}. \]

Not working!

Consider \(z (1 - \frac{1}{z})^{1/2}\). The principal branch of \((1 - \frac{1}{z})^{1/2}\) is

\[ e^{ \frac{1}{2} \log (1 - \frac{1}{z})}. \]

Has cuts where \(1 - \frac{1}{z}\) is negative real, i.e. \(\frac{1}{z}\) is real and \(> 1\). So, the cut is \([-1, 1]\).

\[ w = f(z) = z e^{\frac{1}{2} \log (1 - \frac{1}{z})}. \]