Math 210: Mathematical Methods for Physical Sciences and Engineering

Lecture Notes on Complex Analysis

Plan. 3 – 4 Weeks

1. Analytic Functions
   - Limits and continuity
   - Analiticity. The Cauchy-Riemann Equations
   - Elementary Analytic Functions (trig., exp. log. functions)
2. Complex Integration
   - Definition of Contour Integrals.
   - Cauchy's Integral Theorem. / Formula.
3. Series Representation (for Analytic Functions)
   - Taylor Series / Power Series
   - Laurent Series
   - Zeros and Singularities
4. Residue Theory
   - The Residue Theorem
   - Techniques of Integration
5. Conformal Mapping (If time permits)
   - Concept.
   - Möbius Transformations
   - The Schwarz-Christoffel Transformations
   - Applications.

Section 1. Analytic Functions

Section 1.1. Complex-Valued Functions, Limits, and Continuity

Notation: $\mathbb{R} = \{\text{all real numbers}\}$
$\mathbb{C} = \{\text{all complex numbers}\}$

Complex number $z = a + bi$, $a, b \in \mathbb{R}$

real part: $\operatorname{Re} z = a$.

imaginary part: $\operatorname{Im} z = b$.

$a = \frac{z + \bar{z}}{2}$, $b = \frac{z - \bar{z}}{2i}$

$i^2 = -1$.

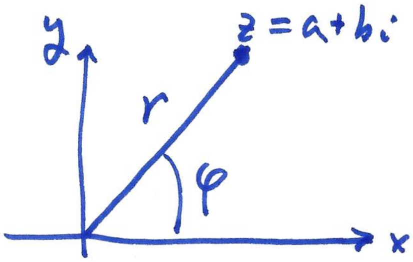
absolute value or modulus: $|z| = \sqrt{a^2 + b^2}$

the complex conjugate: $\bar{z} = a - bi$

$|z|^2 = z\bar{z}$, $|\bar{z}| = |z|$.

$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$.

$|z_1 z_2| = |z_1||z_2|$ $|z_1/z_2| = |z_1|/|z_2|$.



$z = re^{i\varphi}$

$r = |z| = \sqrt{a^2 + b^2}$

$\varphi = \arg z = \arctan \frac{b}{a}$

$\arg i = \frac{\pi}{2} + 2k\pi \quad (k = 0, \pm 1, \pm 2, \cdots)$

$|z_1 \pm z_2| \leq |z_1| + |z_2|$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$$e^z = e^{a+bi} = e^a \cdot e^{bi} = e^a(\cos b + i \sin b)$$

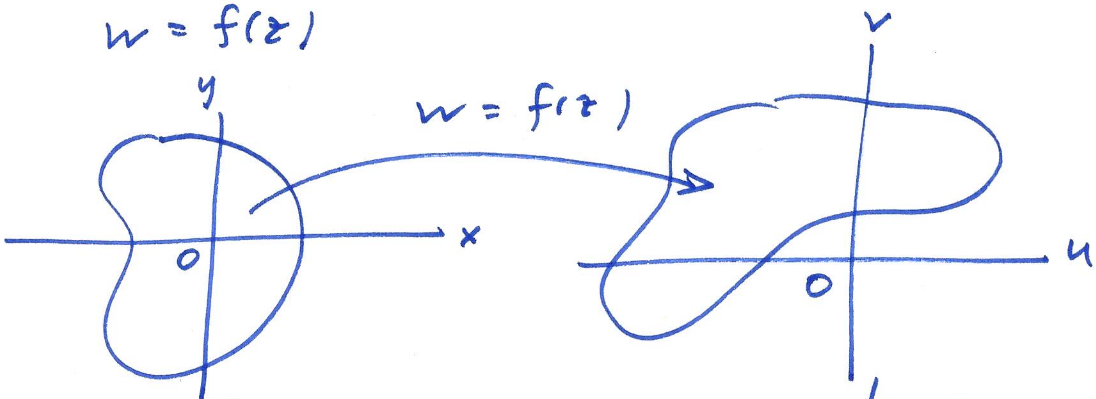
If $m \geq 1$ is an integer, then:

$$1^{1/m} = \left(e^{i2\pi k}\right)^{1/m} = e^{i\frac{2\pi k}{m}} = \cos\frac{2\pi k}{m} + i \sin\frac{2\pi k}{m}$$

$(k = 0, 1, 2, \cdots, m-1)$

Complex function (or: Complex-valued function).

$w = f(z)$



z-plane   w-plane

$$w = u(x,y) + i\,v(x,y), \qquad z = x + iy.$$

Examples: (1) $w = z^2 + 2z = (x+iy)^2 + 2(x+iy)$

$$= x^2 - y^2 + 2xyi + 2x + 2yi$$

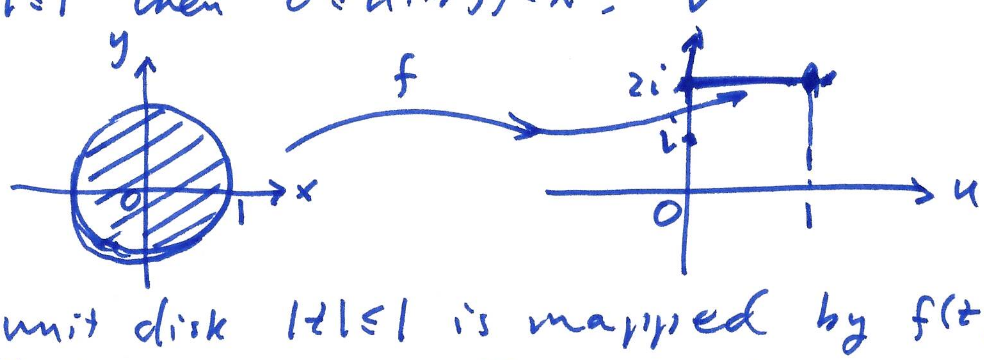
$$= (x^2 - y^2 + 2x) + (2xy + 2y)i$$

$$= \underbrace{\quad}_{u(x,y)} + \underbrace{\quad}_{v(x,y)}\, i$$

(2) $f(z) = x^2 + 2i$ $\qquad \begin{cases} u(x,y) = x^2 \\ v(x,y) = 2 \end{cases}$

$$= \frac{z + \bar{z}}{2} + 2i$$

If $|z| \le 1$ then $0 \le u(x,y) \le x^2$.



The unit disk $|z| \le 1$ is mapped by $f(z) = x^2 + 2i$ to the line segment $[2i, 1+2i]$

(3) $f(z) = z^3$.

Limits

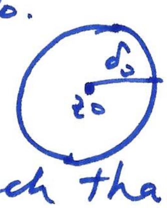
Theorem $z_n = x_n + i y_n \quad (n = 1, 2, \cdots)$

$$\lim_{n\to\infty} z_n = z = x + iy \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Definition For any $\varepsilon > 0$, there exists N (integer), such that $n \geq N \Longrightarrow |z_n - z| < \varepsilon$.

Let $w = f(z)$ defined on $0 < |z - z_0| < \delta_0$.

Definition $\lim_{z\to z_0} f(z) = w_0$



if for any $\varepsilon > 0$, there exists $\delta (\leq \delta_0)$ such that

$0 < |z - z_0| < \delta \Longrightarrow |f(z) - w_0| < \varepsilon$.

Theorem $\lim_{z\to z_0} f(z) = w_0$ $\qquad f(z) = u(x,y) + i\,v(x,y)$

$w_0 = u_0 + i\,v_0$

$z_0 = x_0 + i\,y_0$.

$$\iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$$

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

Definition If $w = f(z)$ is defined on $|z - z_0| < \delta_0$ and $\lim_{z\to z_0} f(z) = f(z_0)$, then $f(z)$ is continuous at z_0.

Theorem $f(z) = u(x,y) + i\,v(x,y)$ is continuous at ~~(x_0, y_0)~~ $z = x_0 + i y_0$ $\iff$ u, v are continuous at (x_0, y_0).

Theorem If $\lim_{z\to z_0} f(z) = \alpha$, $\lim_{z\to z_0} g(z) = \beta$, then

$$\lim_{z\to z_0} [f(z) \pm g(z)] = \alpha \pm \beta,$$

$$\lim_{z\to z_0} f(z)\,g(z) = \alpha\beta. \qquad \lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\alpha}{\beta}$$

(if $\beta \neq 0$)

Section 1.2 Differentiability. The Cauchy-Riemann Equations

Definition Let $f(z)$ be defined in a neighborhood of z_0 (i.e., $|z-z_0|<\delta_0$ for some $\delta_0>0$). The derivative of f at z_0 is

$$f'(z_0) = \frac{df}{dz}(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z},$$

if the limit exists. In this case we say that f is differentiable at z_0.

Examples (1) $f(z)=\bar{z}$, $|z|$, $\mathrm{Re}\, z$, $\mathrm{Im}\, z$ are all continuous but nowhere differentiable!

Consider $f(z)=\bar{z}$.

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{\overline{z_0+\Delta z}-\overline{z_0}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

If $\Delta z = \Delta x \in \mathbb{R}$, then $\overline{\Delta z} = \Delta z$. $\frac{\overline{\Delta z}}{\Delta z} = 1$.

If $\Delta z = i\Delta y \in i\mathbb{R}$, then $\overline{\Delta z} = -\Delta z$. $\frac{\overline{\Delta z}}{\Delta z} = -1$.

So, the limit as $\Delta z \to 0$ does not exist.

(2) $(z^n)' = nz^{n-1}$.

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

$$p'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1$$

$$\frac{(z+\Delta z)^n - z^n}{\Delta z} = \frac{1}{\Delta z}\left[\left(z^n + \binom{n}{1}z^{n-1}\Delta z + \binom{n}{2}z^{n-2}(\Delta z)^2 + \cdots + \binom{n}{n-1}z(\Delta z)^{n-1} + (\Delta z)^n\right) - z^n\right]$$

$$= \frac{1}{\Delta z}\left[\binom{n}{1}z^{n-1}\Delta z + O((\Delta z)^2)\right]$$

$$= nz^{n-1} + O(\Delta z), \quad \text{as } \Delta z \to 0.$$

Operations. $(f\pm g)'(z)$, $(cf)'(z)$, $(fg)'(z) = .$ $\left(\frac{f}{g}\right)'$

chain rule: $(f(g(z)))' = f'(g(z))\, g'(z)$.

The Cauchy-Riemann equations

Theorem Let $f(z) = u(x,y) + i\,v(x,y)$ be differentiable at $z = x+iy$. Then

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$ (The C.-R. equations)

Moreover $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$.

Example (1) check if $f(z) = \operatorname{Re} z = x$ is differentiable or not.

$u(x,y) = x$, $v(x,y) = 0$.

$\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = 0$. No!

(2) Do the same for $f(z) = (x^2+y) + i(y^2-x)$.

$u(x,y) = x^2+y$, $v(x,y) = y^2 - x$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = 2y$.

$\frac{\partial u}{\partial y} = 1$ $\frac{\partial v}{\partial x} = -1$.

C.-R. equations true for $x = y$.

Proof By definition,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists.}$$

$\Delta z = \Delta x + i\Delta y$. Choose $\Delta y = 0$. $\Delta z = \Delta x \in \mathbb{R}$.

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + i\,v(x+\Delta x, y) - u(x,y) - i\,v(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Choose $\Delta z = i\Delta y$ (i.e., $\Delta x = 0$). Then

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

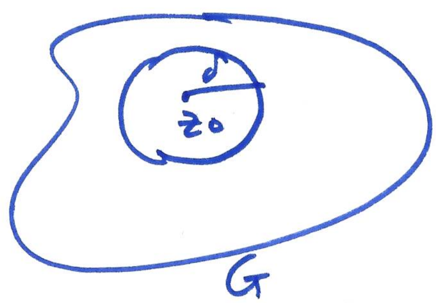
$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Hence, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

and $f'(z) = u_x + i v_x = v_y - i u_y$

$[= u_x - i u_y = v_y + i v_x]$. <u>Q.E.D.</u>

<u>Definition</u> A subset G of $\mathbb{C}$ is open, if for any $z_0 \in G$, there exists a $\delta = \delta(z_0) > 0$ s.t. the disk $\{z : |z - z_0| < \delta\} \subseteq G$. i.e., if G contains a point, then G contains a disk centered at that point.



<u>Definition</u> $w = f(z)$ is analytic in an open set $G \subseteq \mathbb{C}$, if it is differentiable at every point in G.

<u>Example</u> $f(z) = x^2 + y + i(y^2 - x)$

$u_x = 2x$, $v_y = 2y$. $u_y = 1$, $v_x = -1$.

C.-R. equations hold for $x = y$. not open.

So, f is not analytic in any open set.

<u>Theorem</u> Let $f(z) = u(x,y) + i\,v(x,y)$ be defined in an open set G, and $\underset{= x_0 + i y_0}{z_0} \in G$. Suppose

(1) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist in G, and are continuous at (x_0, y_0);

(2) The Cauchy-Riemann equations hold true for f at z_0.

Then, f is differentiable at z_0.

In particular, if all u_x, u_y, v_x, v_y exist and are continuous in G, then $f(z)$ is analytic in G.

<u>Proof</u> $\dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$= \frac{\left[u(x_0+\Delta x, y_0+\Delta y) - u_0(x_0, y_0)\right] + i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)\right]}{\Delta x + i\Delta y}$$

u-part: $u(x_0+\Delta x, y_0+\Delta y) - u_0(x_0, y_0)$

$$= [u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y)] + [u(x_0, y_0+\Delta y) - u(x_0, y_0)]$$

$$= \Delta x\, u_x(x^*, y_0+\Delta y)$$

$$= \Delta x[u_x(x_0, y_0) + o(1)] \quad \text{as } \begin{matrix}\Delta x \to 0 \\ \Delta y \to 0\end{matrix}.$$

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x[u_x + i v_x + o(1)] + \Delta y[u_y + i v_y + o(1)]}{\Delta x + i\Delta y}$$

$$\underset{=}{\overset{\text{C-R.}}{=}} \frac{\Delta x(u_x + i v_x) + i\Delta y(u_x + i v_x)}{\Delta x + i\Delta y} + o(1)$$

$$= u_x + i v_x + o(1). \qquad \text{Q.E.D.}$$

<u>Definition</u> $w=f(z)$ is an <u>entire function</u> if it is analytic in $\mathbb{C}$.

<u>Example</u> $f(z)=e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

$u(x,y) = e^x \cos y$

$v(x,y) = e^x \sin y$.

$u_x = e^x \cos y$. $v_y = e^x \cos y$. $u_x = v_y$

$u_y = -e^x \sin y$. $v_x = e^x \sin y$. $u_y = -v_x$.

So, $f(z)=e^z$ is an entire function.

<u>Theorem</u> If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in an open set G, then both u and v are harmonic in G.

<u>Proof</u> We will know later that in this case u, v are infinitly differentiable. So.

$$u_x = v_y$$

$$\Longrightarrow \quad u_{xx} = v_{yx} = v_{xy} = -u_{yy}.$$

i.e., $\Delta u = u_{xx} + u_{yy} = 0$.

Same for $\Delta v = 0$. <u>Q.E.D.</u>

<u>Application</u>. D: open, connected. f is analytic in D. $f'(z) = 0 \Rightarrow f = \text{const}$.

Section 1.3 Elementary Functions

Exponential function

$$w = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

- $|e^z| = e^x$

  $\arg e^z = y + 2k\pi \quad (k \in \mathbb{Z})$

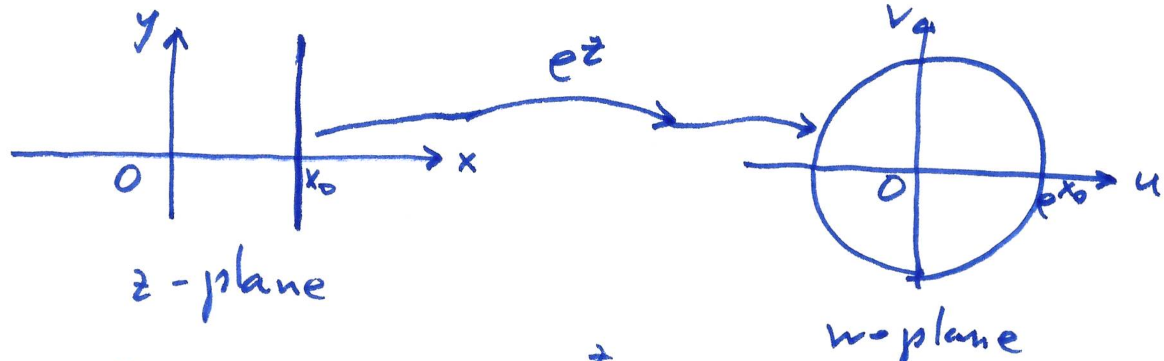
  $\mathbb{Z} = \{\text{all integers}\} = \{0, \pm 1, \pm 2, \cdots\}$

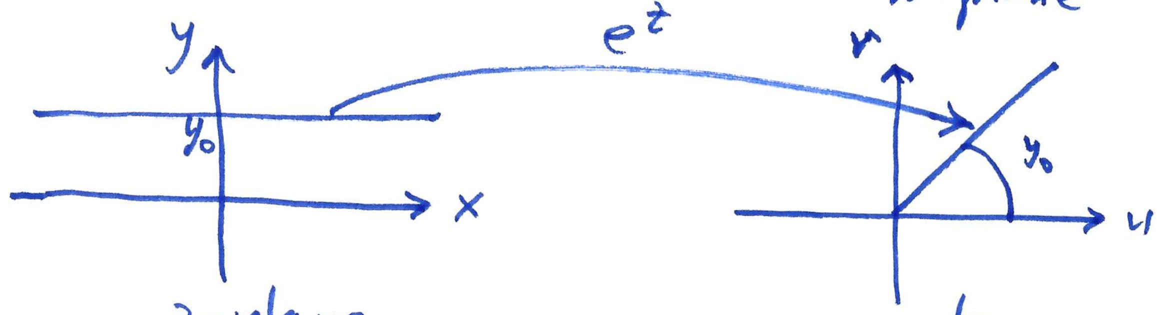
- $e^z = 1 \iff z = 2k\pi i \quad (k \in \mathbb{Z})$

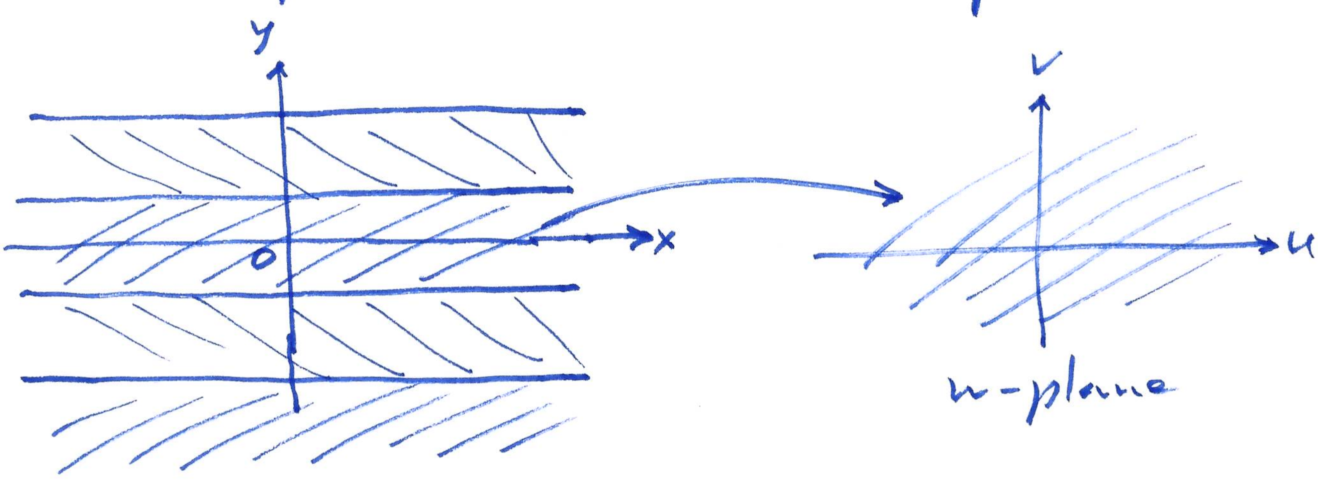
  $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2k\pi i \quad (k \in \mathbb{Z})$

Hence, e^z is $2\pi i$-periodic

- $(e^z)' = e^z$. e^z is an entire function







Trigonometric Functions

Definition: $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$.

$\tan z = \dfrac{\sin z}{\cos z}$, $\cot z = \dfrac{\cos z}{\sin z}$

$\sec z = \dfrac{1}{\cos z}$, $\csc z = \dfrac{1}{\sin z}$

- $(\sin z)' = \cos z$.

  $(\cos z)' = -\sin z$

  $(\tan z)' = \sec^2 z$

  $(\cot z)' = -\csc^2 z$

- $\sin(z+2\pi) = \sin z$

  $\cos(z+2\pi) = \cos z$

  $\sin^2 z + \cos^2 z = 1$

  $\sin(-z) = -\sin z$, $\cos(-z) = \cos z$

  $\cos 2z = \cos^2 z - \sin^2 z$, $\sin 2z = 2\cos z \cos z$

  $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

  $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

> Warning: False: $|\sin z| \le 1$ or $|\cos z| \le 1$ $\forall z \in \mathbb{C}$

Hyperbolic functions

$\sinh z = \dfrac{e^z - e^{-z}}{2}$, $\cosh z = \dfrac{e^z + e^{-z}}{2}$

- $(\sinh z)' = \cosh z$, $(\cosh z)' = \sinh z$
- $\sinh z = -i \sin iz$, $\cosh z = \cos iz$

The Logarithmic Function

Try: $w = \log z$ if $z = e^w$

only for $z \neq 0$ (since $e^w \neq 0 \ \forall w$)

Write $z = re^{i\theta}$, $w = u + iv$. Then $z = e^w \Rightarrow$

$re^{i\theta} = e^u e^{iv}$

$\Rightarrow r = e^u$, i.e. $u = \text{Log}\, r$ Log: the usual logarithmic function.

$e^{i\theta} = e^{iv}$

$v = \theta = \arg z$ <u>multivalued!</u>

<u>Definition</u> If $z \neq 0$ then

$$\log z = \text{Log}|z| + i \arg z$$
$$= \text{Log}|z| + i\,\text{Arg}\, z + i\, 2k\pi$$
$$(k = 0, \pm 1, \pm 2, \cdots)$$

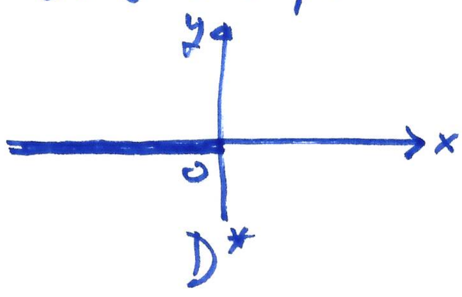
[Arg z: called the principal value of argument, is defined to be the argument such that

$$-\pi < \text{Arg}\, z \leq \pi.]$$

Define: $\text{Log}\, z = \text{Log}|z| + i\,\text{Arg}\, z$

Call it the principal value of the Logarithm $\log z$.

<u>Theorem</u> Let D^* be the domain consisting all $z \in \mathbb{C}$ except those on the negative real axis. Then



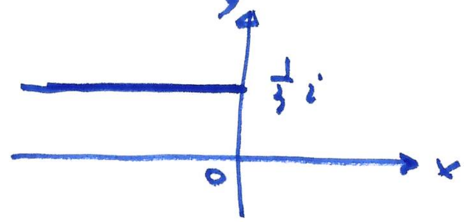
$\text{Log}\, z$ is analytic in D^* and

$$\frac{d}{dz} \text{Log}\, z = \frac{1}{z}, \quad \text{if } z \in D^*.$$

Corollary ① $\operatorname{Arg} z$ is harmonic in D^*.
② $\operatorname{Log}|z|$ is harmonic in $\mathbb{C}\setminus\{0\}$.

Example $f(z) = \operatorname{Log}(3z - i)$

$$f'(z) = \frac{1}{3z-i}\cdot(3z-i)'$$
$$= \frac{3}{3z-i}.$$



A **branch** of $\log z$: Let $\tau \in \mathbb{R}$.

$$\mathcal{L}_\tau(z) = \operatorname{Log}|z| + i\arg_\tau z$$
$$\text{if } \arg_\tau z \in (\tau, \tau + 2\pi].$$

This is a single-valued continuous function.

$$\frac{d}{dz}\mathcal{L}_\tau(z) = \frac{1}{z}.$$

Definition $F(z)$ is a **branch** of a multi-valued function $f(z)$ in a domain D, if $F(z)$ is single-valued and continuous in D, and has the property that, for any $z \in D$, the value of $F(z)$ is one of the values of $f(z)$.

Example $\operatorname{Arg} z$ is a branch of $\arg z$ in D^*
$\operatorname{Log} z$ is a branch of $\log z$ in D^*
$e^{(1/2)\operatorname{Log} z}$ is a branch of $z^{1/2}$ in the right half-plane.

<u>Example</u> Determine a branch of $f(z) = \log(z^3 - 2)$ that is analytic at $z = 0$ and find $f(0)$, $f'(0)$.

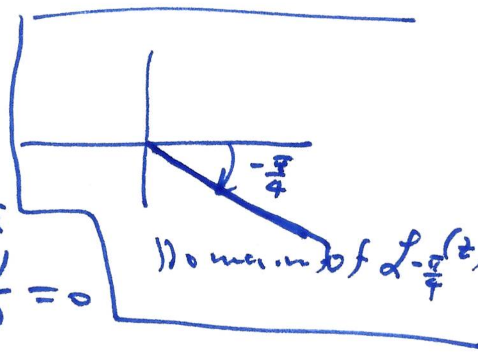
Solution. By the chain rule, it suffices to choose any branch of logarithm that is analytic at $g(0) = -2$ where $g(z) = z^3 - 2$. In particular

$$f(z) = \mathcal{L}_{-\frac{\pi}{4}}(g(z))$$

should work.

$$f(0) = \mathcal{L}_{-\frac{\pi}{4}}(0^3 - 2) = \mathrm{Log}\, 2 + i\pi$$

$$f'(0) = \mathcal{L}'_{-\frac{\pi}{4}}(g(0))\, g'(0) = \frac{g'(0)}{g(0)} = 0$$



<u>Complex Powers Functions</u>

<u>Definition</u> If $\alpha \in \mathbb{C}$, $z \in \mathbb{C}$, $z \neq 0$, then

$$z^\alpha = e^{\alpha \log z}.$$

<u>Example</u> ① $(-2)^i = e^{i \log(-2)} = e^{i\,\mathrm{Log}\, 2}\, e^{-\pi - 2k\pi} \quad (k \in \mathbb{Z})$

as $\log(-2) = \mathrm{Log}\, 2 + (\pi + 2k\pi) i$

② $z^{m/n} = \exp\left(\frac{m}{n} \mathrm{Log}|z|\right) \exp\left(i \frac{m}{n}(\mathrm{Arg}\, z + 2k\pi)\right)$

$(k = 0, 1, \cdots, n-1)$

m, n: integers, $n \geq 1$.

Note: ① If $\alpha \in \mathbb{R}$ (real) z^α is single-valued.

② If $\alpha = \frac{m}{n}$ n, m: integers, then z^α: finitely many values.

③ z^α: infinitely many values for other cases.

<u>Example</u> Define a branch of $(z^2-1)^{1/2}$ that is analytic in the exterior of the unit disk $|z|<1$

Solution. Need to find $w=f(z)$ analytic outside unit circle and

$$w^2 = z^2 - 1.$$

The principal branch of $(z^2-1)^{1/2}$ is

$$e^{(1/2)\,\mathrm{Log}(z^2-1)}.$$

Not working!

Consider $z\left(1-\frac{1}{z^2}\right)^{1/2}$. The principal branch of $\left(1-\frac{1}{z^2}\right)^{1/2}$ is

$$e^{\frac{1}{2}\mathrm{Log}\left(1-\frac{1}{z^2}\right)}$$

has cuts where $1-\frac{1}{z^2}$ is negative real. i.e. $\frac{1}{z^2}$ is real. and >1. So the cut is $[-1,1]$.

$$w = f(z) = z\, e^{\frac{1}{2}\mathrm{Log}\left(1-\frac{1}{z^2}\right)}.$$