Section 2. Complex Integration

Section 2.1 Contour Integrals

Definition A smooth (or regular) arc (or curve) is a function (or the image of ~~the~~ a function) $z : [a,b] \to \mathbb{C}$, for some $a, b \in \mathbb{R}$ with $a < b$, such that

(1) $z \in C^1([a,b]; \mathbb{C})$. i.e., $z = z(t)$, and $z' = z'(t)$ are continuous functions on $[a,b]$;

(2) $z'(t) \neq 0 \quad \forall t \in [a,b]$;

(3) z is one-to-one on $[a,b]$, i.e., $t_1, t_2 \in [a,b]$, $t_1 \neq t_2 \Longrightarrow z(t_1) \neq z(t_2)$.

A closed smooth ~~curve~~ arc (or curve) is a function $z : [a,b] \to \mathbb{C}$ such that

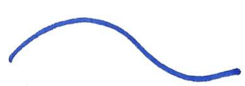
(1) $z \in C^1([a,b]; \mathbb{C})$;

(2) $z'(t) \neq 0 \quad \forall t \in [a,b]$;

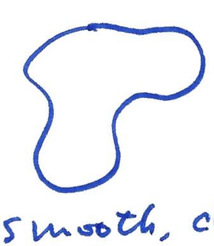
(3)' z is one-to-one on $[a,b)$, and $z(a) = z(b)$

A directed smooth arc (or curve) is a smooth arc together with a specific direction.

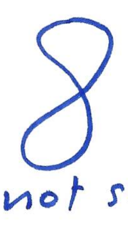
A contour is a head-tail connected, finitely many, directed smooth curves.



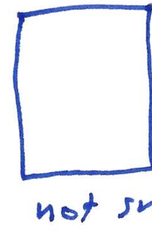
smooth



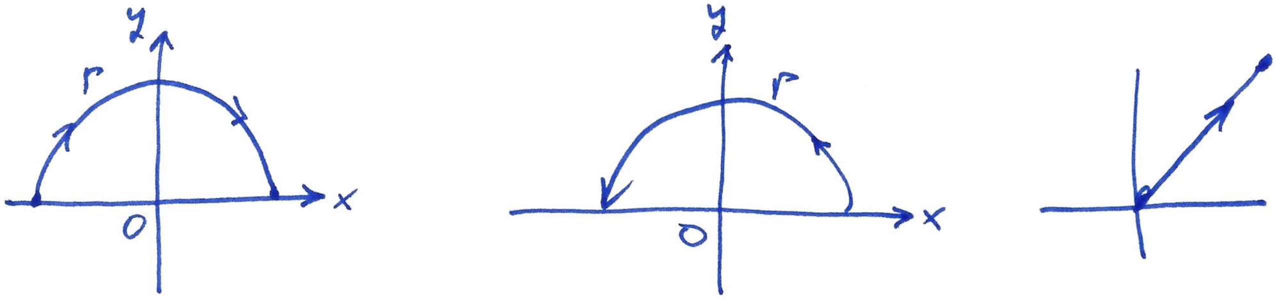
smooth, closed



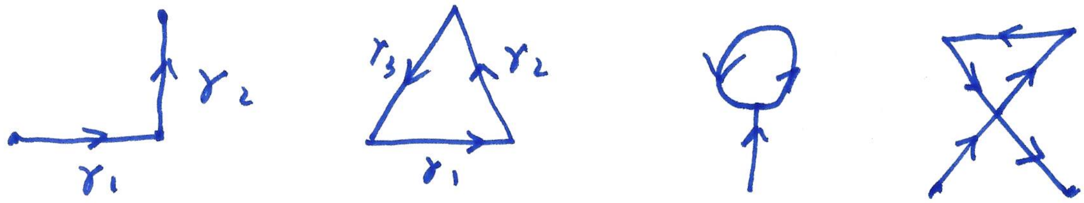
not smooth



not smooth



directed smooth arcs



contours

Length of a smooth arc, $z = z(t)$ $(a \le t \le b)$

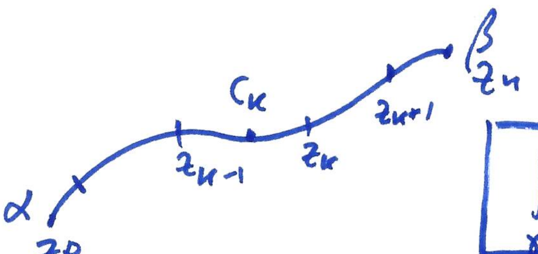
$$= \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt,$$

where $z(t) = x(t) + i\,y(t)$ $(a \le t \le b)$.

<u>Definition of contour integrals</u>

Given: γ: a directed smooth curve (closed or not)

$f : \gamma \to \mathbb{C}$.



The contour integral is defined as

$$\boxed{\int_\gamma f(z)\,dz = \lim_{\substack{\max |\Delta z_k| \to 0 \\ 1 \le k \le n}} \sum_{k=1}^{n} f(c_k)\Delta z_k}$$

if the limit exists. Say f is integrable on γ.

<u>Properties</u>

$$\int_\gamma [f(z) \pm g(z)]\,dz = \int_\gamma f(z)\,dz \pm \int_\gamma g(z)\,dz$$

$$\int_\gamma \alpha f(z)\,dz = \alpha \int_\gamma f(z)\,dz. \quad (\alpha \in \mathbb{C})$$

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz \quad (-\gamma\text{: opposite to } \gamma)$$

Theorem If f is continuous on a directed smooth curve γ, then $\int_\gamma f(z)\,dz$ exists. (and f is integrable on γ.)

Special case $\int_a^b f(t)\,dt$. $(a<b\in\mathbb{R})$.

If $f(t)=u(t)+i\,v(t)$, then $\boxed{\int_a^b f(t)\,dt=\int_a^b u(t)\,dt+i\int_a^b v(t)\,dt}$.

Theorem If ~~f~~ $f(t)$ is continuous on $[a,b]$, then $f(t)$ is integrable on $[a,b]$. Moreover, if $F'(t)=f(t)$ $(a\le t\le b)$ then $\int_a^b f(t)\,dt=F(b)-F(a)$.

Example $\int_0^\pi e^{it}\,dt=\frac{e^{it}}{i}\Big|_0^\pi=\frac{e^{i\pi}}{i}-\frac{e^{i0}}{i}=-\frac{2}{i}=2i$.

Theorem If the directed smooth curve γ is parameterized by $z=z(t)$ $(a\le t\le b)$, and f is a function continuous on γ, then

$$\boxed{\int_\gamma f(z)\,dz=\int_a^b f(z(t))\,z'(t)\,dt}.$$

Sketch of proof

$$\sum_{k=1}^n f(z_k)\,\Delta z_k=\sum_{k=1}^n f(z(t_k))\left[z(t_k)-z(t_{k-1})\right]$$

$$\approx\sum_{k=1}^n f(z(t_k))\,z'(t_k)\,\Delta t_k\longrightarrow\int_a^b f(z(t))\,z'(t)\,dt.$$

Q.E.D.

Note: If both $z=z_1(t)$, $z=z_2(t)$ describe the same γ, then $\int_a^b f(z_1(t))\,z_1'(t)\,dt=\int_a^b f(z_2(t))\,z_2'(t)\,dt$.

<u>Example</u> Let $z_0 \in \mathbb{C}$, $r>0$, and $n \in \mathbb{Z}$. Let C_r denotes the circle $|z-z_0|=r$, traversed once in the counterclockwise direction. Then

$$\int_{C_r} (z-z_0)^n dz = \begin{cases} 0 & \text{if } n \neq -1, \\ 2\pi i & \text{if } n=-1. \end{cases}$$

C_r: $z = z(t) = z_0 + re^{it}$ $(0 \le t \le 2\pi)$.
Let $f(z) = (z-z_0)^n$. Then

$$I \equiv \int_{C_r} (z-z_0)^n dz = \int_0^{2\pi} f(z(t))\, z'(t)\, dt$$

$$= \int_0^{2\pi} (re^{it})^n \, re^{it} \cdot i \, dt$$

$$= i r^{n+1} \int_0^{2\pi} e^{(n+1)it} dt$$

If $n \neq -1$, then $I = i r^{n+1} \frac{1}{(n+1)i} e^{(n+1)it}\Big|_0^{2\pi} = 0$.

If $n=-1$, then $I = i r^{n+1} \cdot 2\pi = i2\pi$.

<u>Definition</u> If γ is a contour consisting of the directed smooth curves $\gamma_1, \cdots, \gamma_n$, and f is continuous on γ then

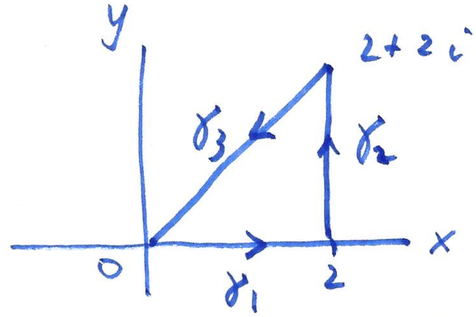
$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz.$$

If γ consists of a single pt, then $\int_\gamma f(z)dz = 0$.

Example $\Gamma: |z-z_0|=r$, traversed twice counterclockwise.

$$\int_\Gamma \frac{dz}{|z-z_0|} = \int_{C_r} \frac{dz}{|z-z_0|} + \int_{C_r} \frac{dz}{|z-z_0|} = 4\pi i.$$

Example



$$\int_\Gamma \bar{z}^2 dz = \int_{\gamma_1} \bar{z}^2 dz + \int_{\gamma_2} \bar{z}^2 dz + \int_{\gamma_3} \bar{z}^2 dz$$

$\gamma_1 : \ z(t) = t \quad (0 \le t \le 2) \qquad z'(t) = 1$

$\gamma_2 : \ z(t) = 2 + ti \quad (0 \le t \le 2), \quad z'(t) = i.$

$\gamma_3 : \ z(t) = -t(1+i) \ (-2 \le t \le 0), \quad z'(t) = -(1+i).$

$$\int_{\gamma_1} \bar{z}^2 dz = \int_0^2 \bar{t}^2 \cdot 1 \, dt = \int_0^2 t^2 dt = \frac{8}{3}$$

$$\int_{\gamma_2} \bar{z}^2 dz = \int_0^2 (2-ti)^2 i \, dt = -\frac{(2-2i)^3}{3} + \frac{8}{3} = \dots$$

$$\int_{\gamma_3} \dots$$

Finally, $\displaystyle\int_\Gamma \bar{z}^2 dz = \frac{16}{3} + \frac{32i}{3}.$

Independence of Path

Theorem If $F'(z)=f(z)$ is continuous in a connected open set $D \subseteq \mathbb{C}$ and Γ is a contour in D with initial point z_I and terminal point z_T, then

$$\int_\Gamma f(z)\,dz = F(z_T)-F(z_I).$$

[Generalization of the Fundamental Thm of the calculus.]

In particular, if Γ is closed (a loop), then

$$\int_\Gamma f(z)\,dz = 0.$$

Proof $\displaystyle\int_\Gamma f(z)\,dz = \sum_{j=1}^{n}\int_{\gamma_j} f(z)\,dz = \sum_{j=1}^{n}\int_{\tau_{j-1}}^{\tau_j} f(z(t))\,z'(t)\,dt$

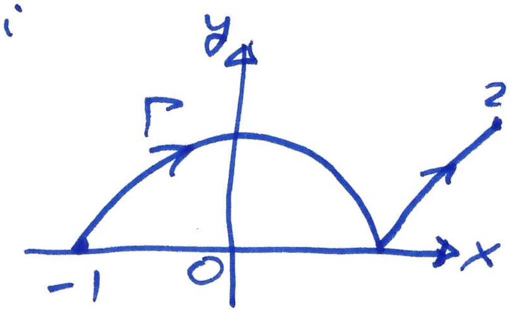
$$= \sum_{j=1}^{n}\int_{\tau_{j-1}}^{\tau_j}\frac{d}{dt}\big[F(z(t))\big]\,dt = \sum_{j=1}^{n} F(z(\tau_j))-F(z(\tau_{j-1}))$$

$$= F(z(\tau_n))-F(z(\tau_0)) = F(z_T)-F(z_I).$$ Q.E.D.

Example $\displaystyle\int_\Gamma \cos z\,dz = \sin z\Big|_{-1}^{2+i}$

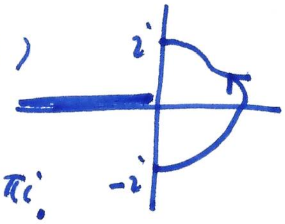
$= \sin(2+i)-\sin(-1)$

$= \sin(2+i)+\sin 1.$

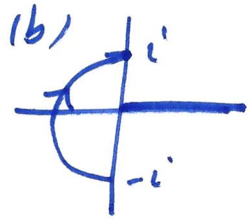


Example $\displaystyle\int_\Gamma \frac{1}{z}\,dz = ?$

(a)



(b)



(a) $\displaystyle\int_\Gamma \frac{dz}{z} = \operatorname{Log} z\Big|_{-i}^{i} = \frac{\pi}{2}i-\left(-\frac{\pi}{2}i\right) = \pi i.$

(b) $\displaystyle\int_\Gamma \frac{dz}{z} = \mathcal{L}_0(z)\Big|_{-i}^{i} = \frac{\pi}{2}i-\frac{3\pi}{2}i = -\pi i.$

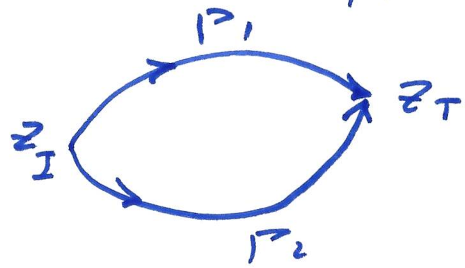
Theorem Let f be continuous in a connected open set D. Then the following are equivalent:

(1) $f(z)$ has an antiderivative in D: $f(z) = F'(z)$.

(2) $\int_\Gamma f(z)\,dz = 0$ for any loop (closed contour) in D.

(3) The contour integrals of f are independent of path in D. i.e., if Γ_1, Γ_2 are two contours in D with same initial and terminal points, then $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$.

Why $(2) \Longleftrightarrow (3)$?



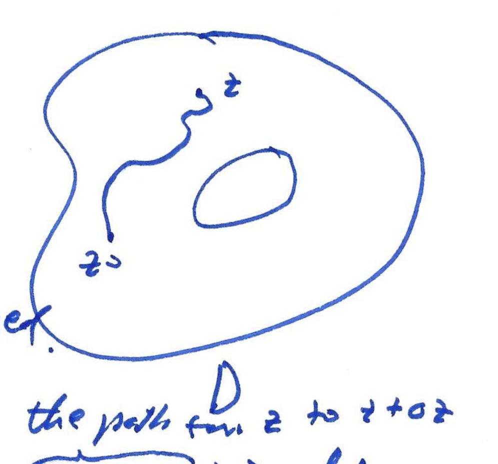
$\Gamma = \Gamma_2 + (-\Gamma_1)$ is a loop.

$$\int_\Gamma f(z)\,dz = 0 \Longleftrightarrow \int_{\Gamma_2} f(z)\,dz + \int_{-\Gamma_1} f(z)\,dz = 0$$

But: $\int_{-\Gamma_1} = -\int_{\Gamma_1}$. So, $\int_{\Gamma_2} f\,dz = \int_{\Gamma_1} f\,dz$.

$(1) \Rightarrow (2)$ By the previous Theorem.

Pf Only $(2) \xcancel{\Rightarrow} (1)$, $+(3)$.



Fix $z_0 \in D$. $\forall z \in D$. Pick up a path $\Gamma: z_0 \to z$. So, $(1)+(2)$

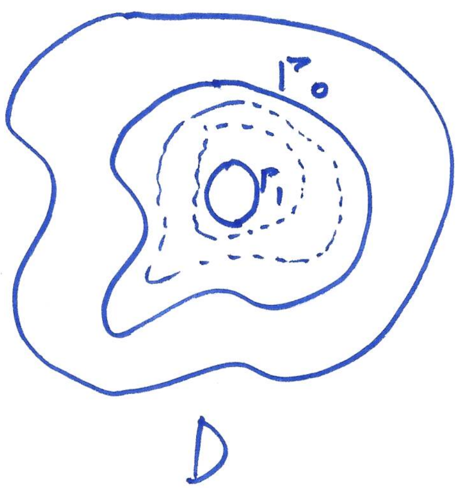
$\Rightarrow$ $F(z) = \int_\Gamma f(z)\,dz$ is well-defined.

i.e., indep. on the path.

Now,
$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\int_0^1 f(\overbrace{z + t\Delta z}^{\text{the path from } z \text{ to } z+\Delta z})\,\Delta z\,dt$$
$$= \int_0^1 f(z + t\Delta z)\,dt \to f(z) \text{ as } \Delta z \to 0. \quad \underline{\text{Q.E.D.}}$$

Section 2.2 Cauchy's Integral Theorem

A **domain** = an open set that is connected.



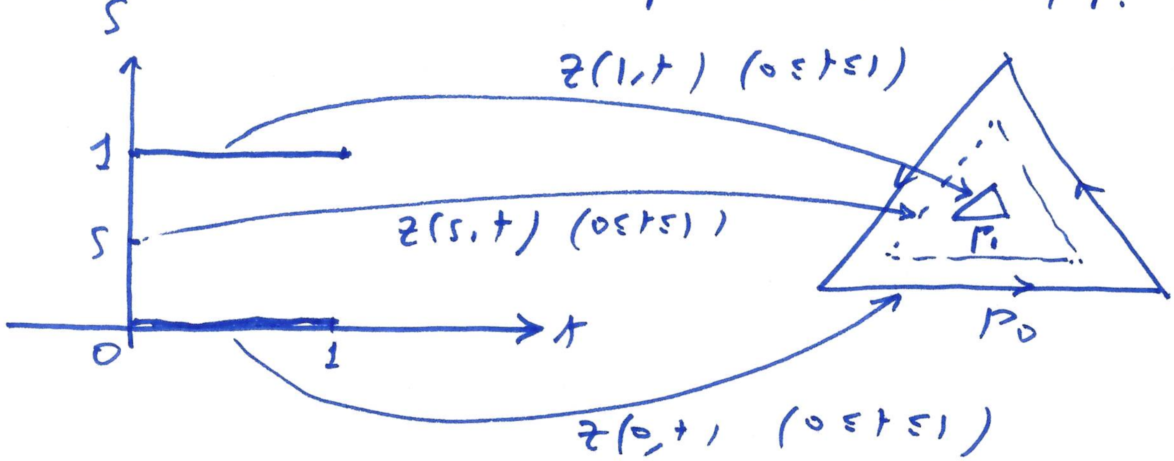
Let Γ_0, Γ_1 be two loops inside a domain $D \subseteq \mathbb{C}$. We say that Γ_0 can be continuously deformed to Γ_1 if there is a family of loops, $\{\gamma_s\}_{0 \le s \le 1}$, such that $\gamma_0 = \Gamma_0$, $\gamma_1 = \Gamma_1$ and $\{\gamma_s\}$ is continuous in s.

More precisely, we have

Definition The loop Γ_0 is continuously deformable (or homotopic) to the loop Γ_1 in the domain D if there exists a continuous function $z = z(s,t)$ $(0 \le s \le 1,\ 0 \le t \le 1)$ such that

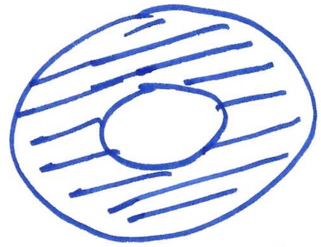
(1) For each $s \in [0,1]$, $z = z(s,t)$ $(0 \le t \le 1)$ parametrizes a loop in D;

(2) $z(0,t)$ $(0 \le t \le 1)$ parametrizes Γ_0, and $z(1,t)$ $(0 \le t \le 1)$ parametrizes Γ_1.

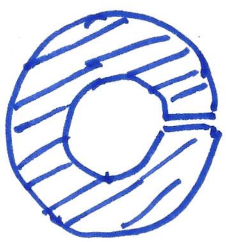


<u>Definition</u> A domain D is <u>simply</u> <u>connected</u>, if any loop in D can be continuously deformed in D to a point in D.

simply connected



not simply connected.



simply connected.

<u>Theorem</u> (Deformation Invariance) Let D be a domain in $\mathbb{C}$, $f : D \to \mathbb{C}$ analytic in D, and Γ_0, Γ_1 two loops in D. If Γ_0 is continuously deformable to Γ_1 in D, then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz .$$

<u>Ideas of proof</u> $\Gamma_0 \to \Gamma_1$. $z = z(s,t)$. Assume $z(s,t)$ has cont. partial derivatives.

$\Gamma_0 : z = z(0,t)$. $\Gamma_1 : z = z(1,t)$.

$\Gamma_s : z = z(s,t)$.

Let $I(s) = \int_{\Gamma_s} f(z)\,dz = \int_0^1 f(z(s,t)) \dfrac{\partial z(s,t)}{\partial t}\,dt$

Then $I'(s) = \int_0^1 \left[f'(z(s,t)) \dfrac{\partial z}{\partial s} \dfrac{\partial z}{\partial t} + f(z(s,t)) \dfrac{\partial^2 z}{\partial s\,\partial t} \right] dt$

$= \int_0^1 \dfrac{\partial}{\partial t} \left[f(z(s,t)) \dfrac{\partial z}{\partial s} \right] dt$

$$= f(z(s,1))\frac{\partial z}{\partial s}(s,1) - f(z(s,0))\frac{\partial z}{\partial s}(s,0)$$

$= 0$ since Γ_s is a loop: $z(s,1) = z(s,0)$, $\frac{\partial z}{\partial s}(s,1) = \frac{\partial z}{\partial s}(s,0)$.

Hence $I(s) = \text{const.}$ $I(1) = I(0)$. Q.E.D.

Cauchy's Integration Theorem Suppose f is analytic in a simply connected domain D and Γ is any loop (closed contour) in D. Then

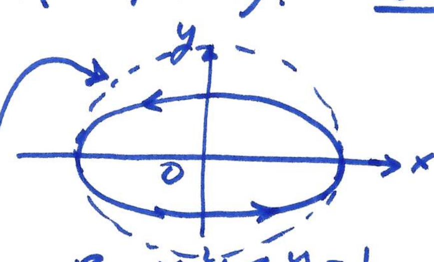
$$\int_\Gamma f(z)\,dz = 0$$

Pf Γ can be deformed continuously in D to a point. Q.E.D.

Thm. In a simply connected domain, any analytic function has an antiderivative, its contour integrals are independent of path, and its loop integrals vanish.

Pf By Cauchy's integration thm and a thm on indep. of paths. (cf. p. $\boxed{22}$). Q.E.D.

Example $\int_\Gamma \frac{1}{z}\,dz$.



$D = \mathbb{C} \setminus \{0\}$. $\frac{1}{z}$ is analytic in D.

Γ: $x^2 + 4y^2 = 1$. traversed positively once.

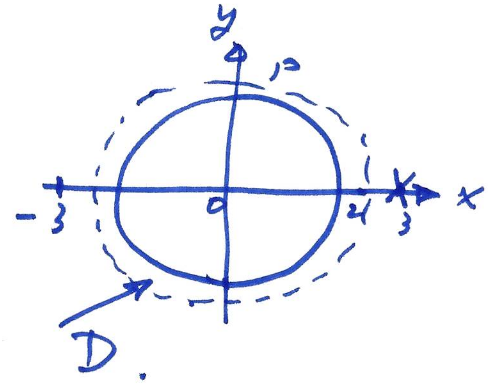
$\Gamma \overset{D}{\equiv} \Gamma_0$. Γ_0: circle.

means Γ is cont. deformable to Γ_0 in D

So, $\int_\Gamma \frac{1}{z}\,dz = \int_{\Gamma_0} \frac{1}{z}\,dz = 2\pi i$.

Example

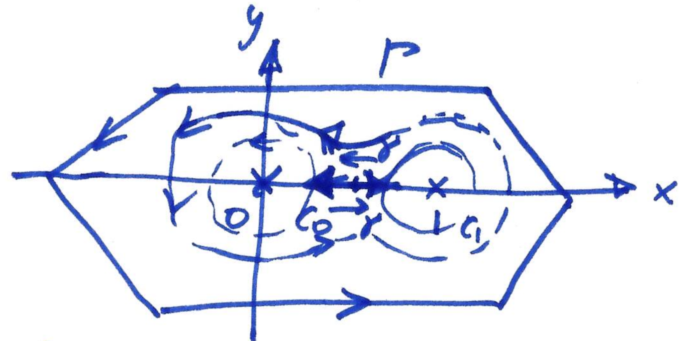
$$\oint_{\Gamma: |z|=2} \frac{e^z}{z^2-9}\,dz = 0$$



Example Let $a \in \mathbb{C}$. Let Γ be any circle not passing a, traversed once in the counterclockwise direction. Then

$$\int_\Gamma \frac{dz}{z-a} = \begin{cases} 0 & \text{if } a \text{ lies } \cancel{\text{inside}} \text{ outside } C. \\ 2\pi i & \text{if } a \text{ lies inside } C. \end{cases}$$

Example $\displaystyle\int_\Gamma \frac{3z-2}{z^2-z}\,dz$



singularities: $z=0$, $z=1$.

C_0: small circle centered at 0   traverse once

C_1: small circle centered at 1.

$\Gamma \xrightarrow{D} C_0 \cup C_1 \cup \gamma \cup \{-\gamma\}$ $\int_{-\gamma} = -\int_\gamma$

So, $\displaystyle\int_\Gamma \frac{3z-2}{z^2-z}\,dz = \left(\int_{C_0} + \int_{C_1}\right) \frac{3z-2}{z^2-z}\,dz \Rightarrow \int_{\gamma \cup \{-\gamma\}} = 0$.

$$\frac{3z-2}{z^2-z} = \frac{3z-2}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

$A(z-1) + Bz = 3z-2$. $(A+B)z - A = 3z - 2$

$A+B = 3$. $-A = -2$.

$A = 2$, $B = 1$.

$$\int_\Gamma \frac{3z-2}{z^2-z}\,dz = \int_{C_0}\left(\frac{2}{z}+\frac{1}{z-1}\right)dz + \int_{C_1}\left(\frac{2}{z}+\frac{1}{z-1}\right)dz$$

$$= \int_{C_0}\frac{2}{z}\,dz + \int_{C_0}\frac{1}{z-1}\,dz + \int_{C_1}\frac{2}{z}\,dz + \int_{C_1}\frac{1}{z-1}\,dz$$

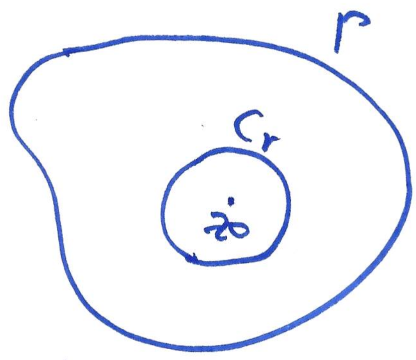
$$= 2\cdot 2\pi i + 0 + 0 + 2\pi i = \boxed{6\pi i}.$$

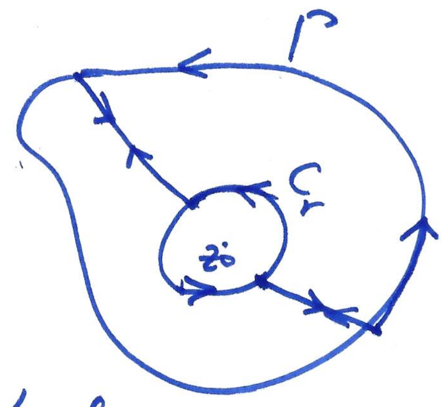
Section 2.3 Cauchy's Integral Formula

Thm (Cauchy's Integral Formula) Let f be analytic in a simply connected domain D. Let Γ be a closed contour (i.e. loops), traversed in the counterclockwise direction once, in D. Let z_0 be a point inside Γ. Then

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,dz$$

Proof





$$\int_\Gamma \frac{f(z)}{z-z_0}\,dz = \int_{C_r}\frac{f(z)}{z-z_0}\,dz$$

$$= \underbrace{\int_{C_r}\frac{f(z_0)}{z-z_0}\,dz}_{=2\pi i f(z_0)} + \underbrace{\int_{C_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz}_{\to 0 \text{ as } r\to 0^+}.$$

on C_r:

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| = \frac{|f(z)-f(z_0)|}{r} \le \frac{M_r}{r}$$

$$\left| \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| \le \frac{M_r}{r}\cdot 2\pi r = 2\pi M_r.$$

$$M_r = \max_{z\in C_r} |f(z)-f(z_0)| \longrightarrow 0 \quad \text{as } r \to 0^+.$$

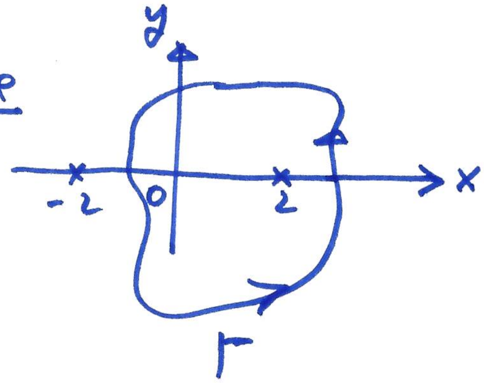
<u>Q.E.D.</u>

<u>Example</u>

$$\oint_{|z-2|=3} \frac{e^z+\sin z}{z}\,dz = 2\pi i\,(e^z+\sin z)\Big|_{z=0}$$

$$= 2\pi i.$$

<u>Example</u>



$$\int_\Gamma \frac{\cos z}{z^2-4}\,dz$$

$$= \int_\Gamma \frac{\frac{\cos z}{z+2}}{z-2}\,dz$$

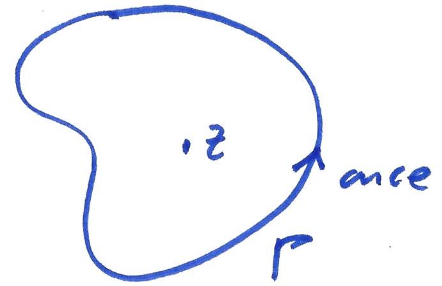
$$= 2\pi i\,\frac{\cos z}{z+2}\Big|_{z=2} = \frac{\pi i \cos 2}{2}.$$

<u>Example</u>

$$\oint_{|z|=1} \frac{z^2 e^z}{2z+i}\,dz = \oint_{|z|=1} \frac{\frac{1}{2} z^2 e^z}{z-\left(-\frac{i}{2}\right)}\,dz$$

$$= 2\pi i\ \frac{1}{2} z^2 e^z\Big|_{z=-\frac{i}{2}} = \frac{\pi i}{4} e^{-i/2}.$$

Cauchy's integral formula is very useful. Here, we use that to show that analytic functions have infinitely many derivatives.



$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Formally.

$$f'(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

$$f''(z) = \frac{2!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta$$

$$\cdots\cdots$$

<u>Theorem</u> If f is analytic in a simply connected domain D, Γ is a loop, traversed once positively in D and z is a point inside Γ, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \quad (n = 1, 2, \cdots).$$

[$n=0$: Cauchy's formula]

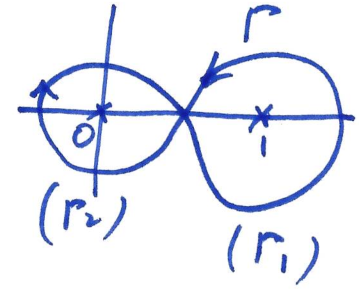
<u>Pf</u>. Only for $n=1$. Use definition. Q.E.D.

<u>Thm</u>. If f is analytic in an ~~domain~~ open set D, then all f', f'', f''', $\cdots$, $f^{(n)}$, $\cdots$ exist and analytic in D.

$f(z) = e^{5z}$, $z_0 = 0$.

<u>Example</u> $$\oint_{|z|=1} \frac{e^{5z}}{z^3} = \frac{2\pi i \cdot f''(0)}{2!} = 25\pi i.$$

<u>Example</u> $$\int_\Gamma \frac{2z+1}{z(z-1)^2}\, dz$$



$$= \int_{\Gamma_1} \frac{\frac{2z+1}{z}}{(z-1)^2}\, dz + \int_{\Gamma_2} \frac{\frac{2z+1}{(z-1)^2}}{z}\, dz$$

$$= 2\pi i \frac{d}{dz}\left(\frac{2z+1}{z}\right)\Big|_{z=1} - 2\pi i \frac{2z+1}{(z-1)^2}\Big|_{z=0} = -2\pi i - 2\pi i = -4\pi i.$$

Section 2.4 Bounds for Analytic Functions

Theorem Let $C_R = \{z \in \mathbb{C} : |z - z_0| = R\}$ $(z_0 \in \mathbb{C}, R > 0)$ be contained in a domain D and $f = f(z)$ is analytic in D. Suppose $\exists M > 0$ such that $|f(z)| \le M$ for all $z \in C_R$. Then

$$\left|f^{(n)}(z_0)\right| \le \frac{n!\,M}{R^n} \quad (n = 1, 2, \cdots).$$

Proof By Cauchy's integral formula,

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i} \oint_{C_R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta\right|$$

$$\le \frac{n!}{2\pi} \max_{\zeta \in C_R} \left|\frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\right| \cdot 2\pi R \le \frac{n!\,M}{R^n}.$$

Q.E.D.

Liouville's Theorem The only bounded entire functions are constant functions.

Proof Take $n = 1$ and send $R \to \infty$: $f' = 0$.
Use the Cauchy-Riemann equations, and $f'(z) = u_x + i v_x = 0$ to get $u, v =$ consts. Q.E.D.

Fundamental Theorem of Algebra Any $\overbrace{\text{non constant}}^{(\text{deg.} \ge 1)}$ polynomial with complex coefficients has at least one zero.

Proof Let $P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.
$(a_j \in \mathbb{C}, j = 0, 1, \cdots, n;\ a_n \ne 0,\ n \ge 1)$.

Suppose $p_n(z) \neq 0 \ \forall z \in \mathbb{C}$. Then $\frac{1}{p(z)}$ is a bounded entire function. Hence $\frac{1}{p(z)} = \text{const}$. This is a contradiction. <u>Q.E.D.</u>

Boundedness of $\frac{1}{p(z)}$ for $p(z) \neq 0$:

$$\left|\frac{1}{p_n(z)}\right| = \frac{1}{|a_n||z|^n\left|1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_0}{a_n z^n}\right|}$$

$\exists A \gg 1$: $|z| \geq A$:

$$\leq \frac{1}{|a_n||z|^n\left(1 - \left|\frac{a_{n-1}}{a_n}\right| \cdot \frac{1}{|z|} - \left|\frac{a_{n-2}}{a_n}\right| \cdot \frac{1}{|z|^2} - \cdots - \frac{|a_0|}{|a_n||z^n|}\right)}$$

$$\leq \frac{1}{|a_n||z|^n\left(1 - \frac{1}{2}\right)}$$

$$\leq \frac{2}{|a_n| A^n}$$

For $|z| \leq A$, $\left|\frac{1}{p(z)}\right| \leq B = \text{const}$. since $\frac{1}{p(z)}$ is continuous and $\{z : |z| \leq A\}$ is compact.

<u>Corollary</u> Any complex polynomial of degree n ($n \geq 1$) has exactly n roots.

<u>Mean-value Thm</u> ~~for any~~ If $f(z)$ is analytic in a domain D containing $\{z \in \mathbb{C} : |z - z_0| \leq R\}$. ($z_0 \in \mathbb{C}$, $R > 0$), then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\, dt$$

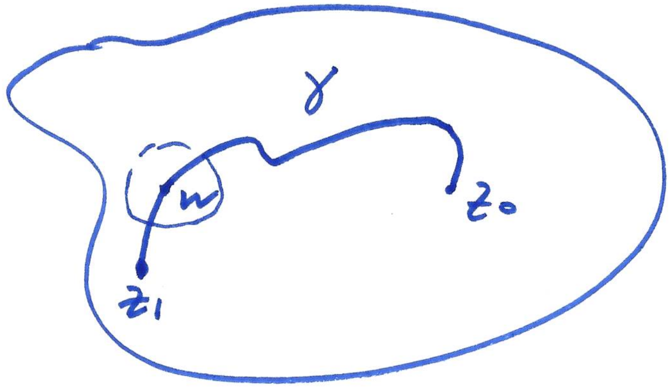
<u>Proof</u> $f(z_0) = \frac{1}{2\pi i}\oint_{C_R} \frac{f(z)}{z - z_0}\, dz \ \underset{C_R : z = z_0 + Re^{it} \, (0 \leq t \leq 2\pi)}{=\!=} \ \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + Re^{it})\, \cancel{iRe^{it}}\, dt}{\cancel{Re^{it}}}$

Q.E.D.

Theorem If $f(z)$ is analytic in a domain D and $|f(z)|$ achieves its maximum value at a point $z_0 \in D$, then $f =$ const. in D.

Pf. Case 1. D = a disk $= \{|z-z_0| < R\}$. So $|f|$ achieves its max. value at the center of D. Mean-Value Thm $\Longrightarrow$ $f(z) = f(z_0)$ $\forall z \in D$.

Case 2. General domain D.



If $\exists z_1 \in D$ such that $|f(z_1)| < |f(z_0)|$, then

① $\exists$ path γ connecting z_0 and z_1, $\gamma \subset D$.

② $\exists$ 1st point w in γ (from z_0 to z_1) such that $|f(z)| = |f(z_0)|$ for all $z \in \gamma$ proceeding w on γ, and $|f(z)| < |f(z_0)|$ for some $z \in \gamma$ after w.

③ A disk $\{z \in \mathbb{C} : |z - w| < R\} \subset D$. Case 1 $\Longrightarrow$ on this disk $f(z) \equiv f(w) = f(z_0)$.

Hence, a contradiction!

So $|f(z)| =$ const. in D.

Hence $u^2 + v^2 =$ consts. But, continuity of u, v and connectivity of D $\Longrightarrow$ $u, v =$ consts. Q.E.D.

Maximum Principle A function analytic in a bounded domain and continuous up to and including its boundary attaints its maximum modulus on the boundary.