Section 2. Complex Integration

Section 2.1 Contour Integrals

Definition: A smooth (or regular) arc \( z \) is a function (or the image of the function) \( z : [a, b] \rightarrow \mathbb{C} \), for some \( a, b \in \mathbb{R} \) with \( a < b \), such that
1. \( z \in C^1([a, b]; \mathbb{C}) \), i.e., \( z = z(t) \) and \( z' = z'(t) \) are continuous functions on \([a, b]\);
2. \( z'(t) \neq 0 \) \( \forall t \in [a, b] \);
3. \( z \) is one-to-one on \([a, b]\), i.e., \( t_1, t_2 \in [a, b] \)
   \( t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2) \).

A closed smooth arc \( \Gamma \) is a function \( z : [a, b] \rightarrow \mathbb{C} \)
such that
1. \( z \in C^1([a, b]; \mathbb{C}) \);
2. \( z'(t) \neq 0 \) \( \forall t \in [a, b] \);
3. \( z \) is one-to-one on \([a, b]\), and \( z(a) = z(b) \).

A directed smooth arc \( \gamma \) is a smooth arc together with a specific direction.

A contour is a head-tail connected, finitely many, directed smooth curves.
Length of a smooth arc \( z = z(t) \) \((a \leq t \leq b)\):

\[
\int_a^b |z'(t)| \, dt = \sqrt{\int_a^b (x'(t))^2 + (y'(t))^2 \, dt},
\]

where \( z(t) = x(t) + i \, y(t) \) \((a \leq t \leq b)\).

**Definition of contour integrals**

Given: \( \gamma \): a directed smooth curve (closed or not)

\( f: \gamma \rightarrow \mathbb{C} \).

The contour integral is defined as

\[
\int_{\gamma} f(z) \, dz = \lim_{n \to \infty} \sum_{k=1}^{n} f(z_k) \Delta z_k,
\]

if the limit exists. Say \( f \) is integrable on \( \gamma \).

**Properties**

- \( \int_{\gamma} [f(z) + g(z)] \, dz = \int_{\gamma} f(z) \, dz + \int_{\gamma} g(z) \, dz \)
- \( \int_{\gamma} df(z) \, dz = \int_{\gamma} f(z) \, dz \) \((a \leq b)\)
- \( \int_{-\gamma} f(z) \, dz = -\int_{\gamma} f(z) \, dz \) \((-\gamma: \text{opposite to } \gamma\))
Theorem. If \( f \) is continuous on a directed smooth curve \( \gamma \), then \( \int_{\gamma} f(z) \, dz \) exists. (and \( f \) is integrable on \( \gamma \).)

Special case \( \int_{a}^{b} f(t) \, dt \). \( (a, b \in \mathbb{R}) \).

If \( f(t) = u(t) + i \, v(t) \), then \( \int_{a}^{b} f(t) \, dt = \int_{a}^{b} u(t) \, dt + i \int_{a}^{b} v(t) \, dt \).

Theorem. If \( f(t) \) is continuous on \([a, b]\), then \( f(t) \) is integrable on \([a, b]\). Moreover, if \( F(t) = \int_{a}^{t} f(s) \, ds \) \( (a \leq t \leq b) \) then \( \int_{a}^{b} f(t) \, dt = F(b) - F(a) \).

Example

\[
\int_{0}^{\pi} e^{it} \, dt = \frac{e^{it}}{i} \bigg|_{0}^{\pi} = \frac{e^{i\pi}}{i} - \frac{e^{i0}}{i} = -\frac{2}{i} = 2i.
\]

Theorem. If the directed smooth curve \( \gamma \) is parametrized by \( z = z(t) \) \( (a \leq t \leq b) \), and \( f \) is a function continuous on \( \gamma \), then

\[
\int_{\gamma} f(z) \, dz = \int_{a}^{b} f(z(t)) \, z'(t) \, dt.
\]

Sketch of proof

\[
\sum_{k=1}^{n} f(z_{k}) \Delta z_{k} = \sum_{k=1}^{n} f(z(t_{k})) \left[ z(t_{k}) - z(t_{k-1}) \right]
\]

\[
\approx \sum_{k=1}^{n} f(z(t_{k})) \, z'(t_{k}) \, \Delta t_{k} \rightarrow \int_{a}^{b} f(z(t)) \, z'(t) \, dt.
\]

\[ \text{Q.E.D.} \]

Note: If both \( z = z(t) \) and \( z = z_{2}(t) \) describe the same \( \gamma \), then

\[
\int_{a}^{b} f(z(t)) \, z'(t) \, dt = \int_{a}^{b} f(z_{2}(t)) \, z_{2}'(t) \, dt.
\]
Example: Let $z_0 \in \mathbb{C}$, $r > 0$, and $n \in \mathbb{Z}$. Let $C_r$ denote the circle $|z - z_0| = r$, traversed once in the counterclockwise direction. Then

$$\int_{C_r} (z - z_0)^n \, dz = \begin{cases} 0 & \text{if } n \neq -1, \\ 2\pi i & \text{if } n = -1. \end{cases}$$

Let $f(t) = (z - z_0)^n$. Then

$$I = \int_{C_r} f(z) \, dz = \int_0^{2\pi} f(z(t)) \dot{z}(t) \, dt$$

$$= \int_0^{2\pi} (re^{it})^n \dot{z}(t) \, dt$$

$$= i r^{n+1} \int_0^{2\pi} e^{(n+1)it} \, dt$$

If $n \neq -1$, then $I = i r^{n+1} \frac{1}{(n+1)i} e^{(n+1)it}\bigg|_0^{2\pi} = 0$.

If $n = -1$, then $I = i r^{n+1} 2\pi = 2\pi i$.

Definition: If $\gamma$ is a contour consisting of the directed smooth curves $\gamma_1, \ldots, \gamma_n$, and $f$ is continuous on $\gamma$ then

$$\int_\gamma f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \cdots + \int_{\gamma_n} f(z) \, dz.$$ 

If $\gamma$ consists of a single pt, then $\int f(z) \, dz = 0$. 
Example \( r : [2-20] = r, \) traversed twice counterclockwise.
\[
\int_r \frac{dz}{z^2} = \int_r \frac{dz}{(z-20)} + \int_r \frac{dz}{(z+20)} = 4\pi i.
\]

Example

\[
\begin{align*}
&\gamma_1 : z(t) = t + j(0 \leq t \leq 2), \quad z(0) = 0, \\
&\gamma_2 : z(t) = 2 + t j (0 \leq t \leq 2), \quad z(0) = i, \\
&\gamma_3 : z(t) = -t(1+j) (2 \leq t \leq 0), \quad z(0) = -j.
\end{align*}
\]

\[
\begin{align*}
\int_{\gamma_1} \overline{z}^2 \, dz &= \int_0^2 \overline{t}^2 \cdot 1 \, dt = \int_0^2 t^2 \, dt = \frac{8}{3}, \\
\int_{\gamma_2} \overline{z}^2 \, dz &= \int_0^2 (2-tj)^2 i \, dt = -\frac{(2-2i)^3}{3} + \frac{8}{3} = \cdots, \\
\int_{\gamma_3} \overline{z}^2 \, dz &= \int_0^2 \cdots
\end{align*}
\]

Finally, \( \int_r \overline{z}^2 \, dz = \frac{16}{3} + \frac{32i}{3}. \)
Independence of Path

Theorem  If \( f^{'}(z) = f(z) \) is continuous in a connected open set \( D \subseteq \mathbb{C} \) and \( \Gamma \) is a contour in \( D \) with initial point \( z_i \) and terminal point \( z_T \), then
\[
\int_{\Gamma} f(z) \, dz = F(z_T) - F(z_i).
\]

In particular, if \( \Gamma \) is closed (a loop), then
\[
\int_{\Gamma} f(z) \, dz = 0.
\]

Proof
\[
\int_{\Gamma} f(z) \, dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z) \, dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z(j+1)) - f(z(j)) \, dz(j)
\]
\[
= \sum_{j=1}^{n} \int_{\Gamma_j} \frac{d}{dz} \left[ F(z(j+1)) \right] \, dz(j) = \sum_{j=1}^{n} F(z(j)) - F(z(j-1))
\]
\[
= F(z_T) - F(z_i) = F(z_T) - F(z_i).
\]
Q.E.D.

Example
\[
\int_{\Gamma} \cos zdz = \sin z \bigg|_{z_i}^{z_T} = \sin (z_T) - \sin (z_i)
\]
\[
= \sin (z_T) + \sin (z_i).
\]

Example
\[
\int_{\Gamma} \frac{1}{z^2} \, dz = ?
\]
\[
(a) \quad \int_{\Gamma} \frac{dz}{z^2} = \log z \bigg|_{z=-i}^{z=i} = \pi i - \pi i = 0
\]
\[
(b) \quad \int_{\Gamma} \frac{dz}{z^2} = \log (z) \bigg|_{z=-i}^{z=i} = \frac{\pi}{2} i - \frac{\pi}{2} i = 0
\]
Theorem: Let $f$ be continuous on a connected open set $D$. Then the following are equivalent:

1. $f(z)$ has an antiderivative in $D$, $F(z) = f(z)$.
2. $\int_C f(z)dz = 0$ for any loop (closed contour) in $D$.
3. The contour integrals of $f$ are independent of path in $D$, i.e., if $P_1$, $P_2$ are two contours in $D$ with same initial and terminal points, then $\int_{P_1} f(z)dz = \int_{P_2} f(z)dz$.

Why (2) $\iff$ (3)?

$P = P_2 + (-P_1)$ is a loop.

$$\int_C f(z)dz = 0 \iff \int_{P_1} f(z)dz + \int_{-P_1} f(z)dz = 0$$

But: $\int_{-P_1} f(z)dz = -\int_{P_1} f(z)dz$. So, $\int_{P_2} f(z)dz = \int_{P_1} f(z)dz$.

(1) $\implies$ (2) by the previous theorem.

PF: Only (2) $\implies$ (1).

Fix $z_0 \in D$. For $z \in D$, pick up a path $P: z_0 \to z$. So, (1) + (2) $\implies F(z) = \int f(z)dz$ is well-defined.

i.e., independent on the path.

Now, $\frac{1}{\alpha^2} [(z+\alpha z) - f(z)] = \frac{1}{\alpha^2} \int_0^\alpha f'(z+\alpha z)dz$.

$$= \int_0^\alpha f'(z+\alpha z)dz \to 0$$

as $\alpha \to 0$. Q.E.D.
Section 2.2  Cauchy's Integral Theorem

A domain = an open set that is connected.

Let $P_0, P_1$ be two loops inside a domain $D \subset \mathbb{C}$. We say that $P_0$ can be continuously deformed to $P_1$, if there is a family of loops, $\{q_s\}_{0 \leq s \leq 1}$, such that $q_0 = P_0$, $q_1 = P_1$, and $\{q_s\}$ is continuous in $s$.

More precisely, we have

**Definition**  The loop $P_0$ is continuously deformable (or homotopic) to the loop $P_1$ in the domain $D$, if there exists a continuous function $z = z(s, t)$ $(0 \leq s \leq 1, 0 \leq t \leq 1)$ such that

1. For each $s \in [0, 1]$, $z(s, t) (0 \leq t \leq 1)$ parametrizes a loop in $D$;
2. $z(s, 0) (0 \leq s \leq 1)$ parametrizes $P_0$, and $z(s, 1) (0 \leq s \leq 1)$ parametrizes $P_1$.
Definition. A domain D is **simply connected** if any loop in D can be continuously deformed in D to a point in D.

\[ \begin{align*} \text{simply connected} & \quad \text{not simply connected} \\ \text{simply connected} & \end{align*} \]

**Theorem (Deformation Invariance)**. Let D be a domain in \( \mathbb{C} \), \( f : D \to \mathbb{C} \) analytic in D, and \( P_0, P_1 \) two loops in D. If \( P_0 \) is \( \alpha \) continuously deformable to \( P_1 \) in D, then

\[ \int_{P_0} f(z) \, dz = \int_{P_1} f(z) \, dz. \]

**Ideas of proof**

\[ P_0 \to P_1, \quad z = z(s, t) \]

\[ P_0 : z = z(0, t), \quad P_1 : z = z(1, t), \]

\[ P_s : z = z(s, t). \]

Let \( I(s) = \int_{P_s} f(z) \, dz = \int_0^1 f(z(s, t)) \frac{\partial z(s, t)}{\partial s} \, dt \)

Then \( I'(s) = \int_0^1 \left[ f(z(s, t)) \frac{\partial^2 z}{\partial s^2} - \frac{\partial^2 f}{\partial s \partial t} + f(z(s, t)) \frac{\partial^2 z}{\partial s \partial t} \right] dt = \int_0^1 \frac{\partial}{\partial t} \left[ f(z(s, t)) \frac{\partial z}{\partial s} \right] dt \)
\[
= f(\alpha'(5.1)) \frac{d}{ds} \begin{pmatrix} 5.1 \end{pmatrix} - f(\alpha'(5.0)) \frac{d}{ds} \begin{pmatrix} 5.0 \end{pmatrix} \\
= 0 \quad \text{since } \alpha' \text{ is a loop: } \alpha'(5.1) = \alpha'(5.0), \quad \frac{d}{ds} \begin{pmatrix} 5.1 \end{pmatrix} = \frac{d}{ds} \begin{pmatrix} 5.0 \end{pmatrix}.
\]
Hence \( I(5) = \text{const.} \quad I(1) = I(0). \quad \Omega \Phi \Phi \Phi \).

**Cauchy's Integration Theorem** Suppose \( f \) is analytic in a simply connected domain \( D \) and \( \gamma \) is any loop (closed contour) in \( D \). Then
\[
\int \gamma f(z) \, dz = 0
\]

If \( \gamma \) can be deformed continuously in \( D \)
to a point. \quad \Omega \Phi \Phi \Phi \.

**Thm.** In a simply connected domain, any analytic function has an antiderivative; its contour integrals are independent of path, and its loop integrals vanish.

If \( \gamma \) by Cauchy's integration thm and a thm on indep. of paths. (cf. P[22]). \quad \Omega \Phi \Phi \Phi \.

**Example**
\[
\int \gamma \frac{dz}{z}. \\
D = \mathbb{C} \setminus \{0\}, \quad \frac{1}{z} \text{ is analytic in } D. \\
\gamma \equiv \Gamma_0. \quad \Gamma_0: \text{circle.} \quad \frac{\phi}{\gamma} \text{ means } \gamma \text{ is cont. deformable to } \Gamma_0 \text{ in } D
\]
\( p: x^2 + 4y = 1, \) traversed positively once.
So, \( \int_{\gamma} \frac{1}{z} \, dz = \int_{\gamma} \frac{1}{t} \, dt = 2\pi i. \)

**Example**

\[
\oint_{|z|=2} \frac{e^z}{z^2-9} \, dz = 0
\]

**Example**

Let \( a \in \mathbb{C} \). Let \( \gamma \) be any circle not passing \( a \), traversed once in the counterclockwise direction. Then

\[
\oint_{\gamma} \frac{dz}{z-a} = \begin{cases} 0 & \text{if } a \text{ lies outside } \gamma \\ 2\pi i & \text{if } a \text{ lies inside } \gamma \end{cases}
\]

**Example**

\[
\oint_{\gamma} \frac{3z-2}{z^2-2} \, dz
\]

Singularities: \( z = 0, \quad z = 1 \).

\( C_0 \): Small circle centered at 0.

\( C_1 \): Small circle centered at 1.

\[ P \rightarrow C_0 \cup C_1 \cup \gamma \cup \text{outside} \]

\[
\oint_{\gamma} \frac{3z-2}{z^2-2} \, dz = \left( \oint_{C_0} + \oint_{C_1} \right) \frac{3z-2}{z^2-2} \, dz \Rightarrow \oint_{\gamma} = 0.
\]

\[
\frac{3z-2}{z^2-2} = \frac{3z-2}{2(z-1)} = \frac{A}{z} + \frac{B}{z-1}.
\]

\[
A(z-1) + Bz = 3z - 2. \quad (A+B)z = A = 3z - 2
\]

\[
A + B = 3, \quad -A = -2.
\]
\[
\int_{C_0} \frac{3z-2}{z^2-1} \, dz = \int_{C_0} \left( \frac{2}{z} + \frac{1}{z-1} \right) \, dz + \int_{C_1} \left( \frac{2}{z} + \frac{1}{z+1} \right) \, dz \\
= \int_{C_0} \frac{2}{z} \, dz + \int_{C_0} \frac{1}{z-1} \, dz + \int_{C_1} \frac{2}{z} \, dz + \int_{C_1} \frac{1}{z+1} \, dz \\
= 2 \cdot 2\pi i + 0 + 0 + 2\pi i = 6\pi i.
\]

Section 2.3  Cauchy's Integral Formula

**Theorem (Cauchy's Integral Formula)** Let \( f \) be analytic in a simply connected domain \( D \). Let \( \Gamma \) be a closed contour (or loop), traversed in the counterclockwise direction once, in \( D \). Let \( z_0 \) be a point inside \( \Gamma \). Then

\[
f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-z_0} \, dz
\]

**Proof**

\[
\int_{\Gamma} \frac{f(z)}{z-z_0} \, dz = \int_{C_r} \frac{f(z)}{z-z_0} \, dz + \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0} \, dz
\]

\[
= 2\pi i f(z_0) + 0 \quad \text{as} \quad r \to 0^+.
\]
on \( C_r \):

\[
\left| \frac{f(z)-f(z_0)}{z-z_0} \right| \leq \frac{M_r}{r}
\]

\[
\left| \frac{1}{C_r} \int \frac{f(z)-f(z_0)}{z-z_0} \, dz \right| \leq \frac{M_r}{r} \cdot 2\pi r = 2\pi M_r.
\]

\[M_r = \max_{z \in C_r} |f(z) - f(z_0)| \to 0 \quad \text{as} \quad r \to 0^+.
\]

\[\text{Q.E.D.}\]

**Example**

\[
\int_{\gamma} \frac{e^z + \sin z}{z} \, dz = 2\pi i \left( e^z + \sin z \right) \bigg|_{z=0}
\]

\[|\gamma| = 3
\]

\[= 2\pi i.
\]

**Example**

\[
\int_{\gamma} \frac{\cos z}{z+2} \, dz = \int_{\gamma} \frac{\cos z}{z-(-2)} \, dz
\]

\[= 2\pi i \cdot \frac{\cos z}{z+2} \bigg|_{z=2} = \frac{\pi i \cos 2}{2}.
\]

**Example**

\[
\int_{|z|=1} \frac{2^z e^z}{2^z + i} \, dz = \int_{|z|=1} \frac{\frac{z-2}{z-(-\frac{1}{2})} e^z}{2} \, dz
\]

\[= 2\pi i \cdot \frac{1}{2} \frac{z^2 e^z}{z-\frac{1}{2}} \bigg|_{z=-\frac{1}{2}} = \frac{\pi i e^{-\frac{1}{4}}}{2}.
\]

Cauchy's integral formula is very useful. Here, we use that to show that analytic functions have infinitely many derivatives.
\[
f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(s)}{s - z} \, ds
\]

Formally,
\[
f'(z) = \frac{1}{2\pi i} \int_{C} \frac{f(s)}{(s - z)^2} \, ds
\]
\[
f''(z) = \frac{2}{2\pi i} \int_{C} \frac{f(s)}{(s - z)^3} \, ds
\]

**Theorem** If \( f \) is analytic in a simply connected domain \( D \), \( \Gamma \) is a loop, traversed once positively, in \( D \), and \( z \) is a point inside \( \Gamma \), then
\[
f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C} \frac{f(s)}{(s - z)^{n+1}} \, ds, \quad (n = 1, 2, \ldots)
\]
[\( n = 0 \): Cauchy's integral formula]

\( \Gamma \) Only for \( n = 1 \). Use definition. \( \text{Q.E.D.} \)

**Thm.** If \( f \) is analytic in an open set \( D \), then all \( f', f'', f''', \ldots \) exist and are analytic in \( D \).

\( f(z) = e^{\frac{z^2}{2}}, \ z_0 = 0 \).

**Example** \( \oint_{C} \frac{e^{\frac{z^2}{2}}}{z} \, dz = 2\pi i \left. \frac{f^{(n)}(z)}{n!} \right|_{z=z_0} = 2\pi i c \).

**Example**
\[
\int_{\Gamma} \frac{z^{2n+1}}{(z-1)^2} \, dz = \int_{\Gamma_1} \frac{2z^{2n+1}}{(z-1)^2} \, dz + \int_{\Gamma_2} \frac{2z^{2n+1}}{(z-1)^2} \, dz
\]
\[
= 2\pi i \left. \frac{d}{dz} \left( \frac{2z^{2n+1}}{(2n+1)z} \right) \right|_{z=1} - 2\pi i \left. \frac{2z^{2n+1}}{(2n+1)(z-1)^2} \right|_{z=0} = -2\pi i - 2\pi i = -4\pi i.
\]
Section 3.4 Bounds for Analytic Functions

**Theorem** Let \( CR = \{ z \in \mathbb{C} : 12 - 20 = R \} \) be contained in a domain \( D \) and \( f = f(z) \) is analytic in \( D \). Suppose \( \exists M > 0 \) such that \( |f(z)| \leq M \) for all \( z \in CR \). Then

\[
|f^{(n)}(z_0)| \leq \frac{n! M}{R^n} \quad (n = 1, 2, \ldots)
\]

**Proof** By Cauchy's integral formula,

\[
|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_{C_R} \frac{f(s)}{(s - z_0)^{n+1}} \, ds \right|
\]

\[
\leq \frac{n!}{2\pi} \max_{z \in CR} \left| \frac{f(s)}{(s - z_0)^{n+1}} \right| \cdot 2\pi R \leq \frac{n! M}{R^n} \quad \text{Q.E.D.}
\]

**Liouville's Theorem** The only bounded entire functions are constant functions.

**Proof** Take \( n = 1 \) and send \( R \to \infty \), \( f' = 0 \).

Use the Cauchy–Riemann equations, and \( f(z) = u + iv = 0 \) to get \( u, v = \text{constants} \). \( \text{Q.E.D.} \quad (\deg \geq 1) \)

**Fundamental Theorem of Algebra** Any non constant polynomial with complex coefficients has at least one zero.

**Proof** Let \( p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \).

\((a_i \in \mathbb{C}, i = 0, 1, \ldots, n, \forall a_n \neq 0, n \geq 1)\).
Suppose \( f(z) \neq 0 \) \( \forall z \in \mathbb{C} \). Then \( \frac{1}{f(z)} \) is a bounded entire function. Hence \( \frac{1}{f(z)} = \text{const.} \). This is a contradiction. \( \therefore \).

**Boundedness of \( \frac{1}{f(z)} \) for \( f(z) \neq 0 \):**

\[
\left| \frac{1}{f(z)} \right| = \left| \frac{1}{a_n|z|^n} \right| \left( 1 + \frac{a_{n-1}}{a_n z^{n-1}} + \cdots + \frac{a_0}{a_n z^n} \right) \leq \frac{1}{|a_n| |z|^n} \left| 1 - \frac{A}{|z|^n} \right| \leq \frac{1}{|a_n| |z|^n} (1 - \frac{A}{|z|^n}) \leq \frac{1}{|a_n| A^n} (1 - \frac{1}{2}) \leq \frac{1}{|a_n| A^n} .
\]

For \( |z| \leq A \), \( \left| \frac{1}{f(z)} \right| \leq B = \text{const.} \), since \( \frac{1}{f(z)} \) is continuous and \( \mathbb{C} \) is compact.

**Corollary:** Any complex polynomial of degree \( n (n \geq 1) \) has exactly \( n \) roots.

**Mean-Value Thm for Analytic in a Domain:** If \( f(z) \) is analytic in a domain \( D \) containing \( \{ z \in \mathbb{C} : |z| < R \} \) (\( R > 0 \)), then

\[
f(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z - z_0} \, dz,
\]

**Proof:**

\[
f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z - z_0} \, dz
\]

\( \therefore \)
**Theorem** If \( f(z) \) is analytic in a domain \( D \) and \( |f(z)| \) achieves its maximum value at a point \( z_0 \in D \), then \( f = \text{const.} \) in \( D \).

**PF.**

**Case 1.** \( D = \) a disk = \( \{ |z - z_0| < R \} \). So, \( |f| \) achieves its max. value at the center of \( D \).

Mean-Value Thm \( \Rightarrow f(z) = f(z_0) \quad \forall z \in D \).

**Case 2.** General domain \( D \).

Let \( \exists z_1, \ldots, z_2 \in D \) such that \( |f(z_1)| < |f(z_0)| \), then:

1. \( \exists \) path \( \gamma \) connecting \( z_0 \) and \( z_1 \), \( \forall z \in \gamma \).
2. \( \exists \) 1st point \( w \) in \( \gamma \) (from \( z_0 \) to \( z_1 \)) such that \( |f(z_2)| = |f(z_1)| \) for all \( z \in \gamma \) preceding \( w \) and \( |f(z)| < |f(z_0)| \) for some \( z \in \gamma \) after \( w \).

A disk \( \{ z \in \mathbb{C}: |z - w| < R \} \subset D \). Case 1 \( \Rightarrow \) on this disk \( f(w) = f(z_0) \).

Hence, a contradiction!

So, \( |f(z)| = \text{const.} \in D \).

Hence \( u \) and \( v \) are const.

But, continuity of \( u, v \) and connectivity of \( D \) \( \Rightarrow \) \( u, v = \text{const.} \). \( \Box \).

**Maximum Principle** A function analytic in a bounded domain and continuous up to and including its boundary attains its maximum modulus on the boundary.