Section 3 Series Representations for Analytic Functions

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Section 3.1 Power Series

Let $f(z)$ be analytic in the disk $|z-z_0|<R$.
All $f^{(n)}(z_0)$ exist $(n=0,1,2,\dots)$. Call

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j$$ the Taylor series for f at z_0. If $z_0=0$, call it also the Maclaurin series ~~at~~ for f.

<u>Thm</u> Suppose $f(z)$ is analytic in the disk $|z-z_0|<R$. Then the Taylor series of $f(z)$ at z_0 converges in this disk. Moreover, this convergence is uniform in $|z-z_0|\le R'$ for any $R'\in(0,R)$.

<u>Idea of Proof</u> $$f(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}\,d\zeta$$

$$\frac{1}{\zeta-z}=\frac{1}{(\zeta-z_0)-(z-z_0)}=\frac{1}{\zeta-z_0}\cdot\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}$$

$$=\frac{1}{\zeta-z_0}\left[1+\frac{z-z_0}{\zeta-z_0}+\frac{(z-z_0)^2}{(\zeta-z_0)^2}+\dots+\frac{(z-z_0)^n}{(\zeta-z_0)^n}\right.$$

$$\left.+\frac{\frac{(z-z_0)^{n+1}}{(\zeta-z_0)^{n+1}}}{1-\frac{z-z_0}{\zeta-z_0}}\right].$$

Hence

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0}\,d\zeta + \frac{z-z_0}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^2}\,d\zeta$$

$$+ \cdots + \frac{(z-z_0)^n}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta + T_n(z)$$

Cauchy's formula

$$= f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + T_n(z).$$

$$|T_n(z)| \le \frac{1}{2\pi} \max_{\zeta\in C} |f(\zeta)| \frac{2}{R-R'}\left(\frac{2R'}{R+R'}\right)^{n+1} 2\pi \frac{R+R'}{2}.$$

$\to 0.$ Q.E.D.

<u>Examples</u> (a) $\operatorname{Log} z = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!}(z-1)^j, \quad |z-1|<1.$

(b) $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j, \quad |z|<1.$

(c) $e^z = \sum_0^{\infty} \frac{z^j}{j!} \quad \forall z \in \mathbb{C}.$

(d) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \quad \forall z \in \mathbb{C}.$

(e) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad \forall z \in \mathbb{C}.$

Usual properties.

$$f(z) = \sum_{j=0}^{n} \frac{f^{(n)}(z_0)}{j!}(z-z_0)^n. \Longrightarrow c f(z) = \sum \frac{c f^{(n)}(z_0)}{j!}(z-z_0)^n.$$

$$\left.\begin{array}{l} f(z) = \sum a_n (z-z_0)^n \\ g(z) = \sum b_n (z-z_0)^n \end{array}\right. \Longrightarrow f(z) \pm g(z) = \sum (a_n \pm b_n)(z-z_0)^n$$

smaller R.

$$f(z)g(z) = \sum_0^{\infty} c_n (z-z_0)^n, \quad c_n = \sum_{k=0}^{n} a_{n-k} b_k.$$

<u>Power series</u> $\sum_{n=0}^{\infty} a_n (z-z_0)^n. \quad |z-z_0|<R.$

R: radius of convergence

$$f(z) = \sum a_n (z-z_0)^n, \quad |z-z_0|<R$$

analytic function.

Term-by-term differentiation, integration, etc.

Section 3.2 Laurent Series

Theorem Let $f(z)$ be analytic in an open annulus $r < |z - z_0| < R$. $(0 \le r < R \le \infty)$. Then

$$(*) \quad f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}$$
$$= \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j \qquad (r < |z - z_0| < R).$$

where
$$a_j = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \qquad (j = 0, \pm 1, \pm 2, \cdots),$$

C is any positively oriented simple loop in $r < |z - z_0| < R$, enclosing z_0.

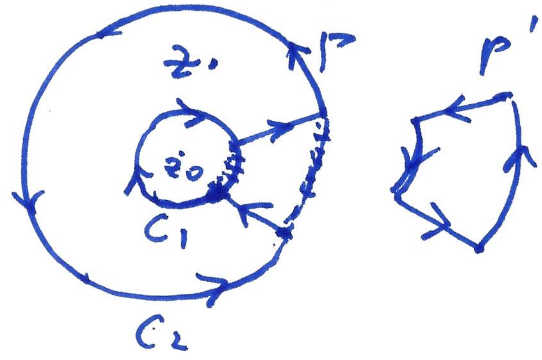
Moreover, the convergence is uniform in $r < \rho_1 \le |z - z_0| \le \rho_2 < R$.

We call $(*)$ the Laurent series for f at z_0 in $r < |z - z_0| < R$.

Ideas of Proof: step 1: Given z. We have

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$
$$0 = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$\Rightarrow \boxed{f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta}$$



step 2: For $\zeta \in C_2$, $|\zeta - z| < 1$. So, previous argument works:

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\sum_{j=0}^{\infty}a_j(z-z_0)^j.$$

Step 3. Consider $\zeta\in C_1$: $|\zeta-z|>1$.

$$\frac{1}{\zeta-z}=\frac{1}{(\zeta-z_0)-(z-z_0)}=-\frac{1}{z-z_0}\cdot\frac{1}{1-\frac{\zeta-z_0}{z-z_0}}.$$

$$=-\frac{1}{z-z_0}\left[1+\frac{\zeta-z_0}{z-z_0}+\frac{(\zeta-z_0)^2}{(z-z_0)^2}+\cdots+\frac{(\zeta-z_0)^n}{(z-z_0)^n}\right.$$

$$\left.+\frac{(\zeta-z_0/z-z_0)^{n+1}}{1-\frac{\zeta-z_0}{z-z_0}}\right].$$

Then, integrate. Q.E.D.

Example $f(z)=\dfrac{z^2-2z+3}{z-2}$. $|z-1|>1$.

Find the Laurent ~~re~~series.

Solution. $\dfrac{1}{z-2}=\dfrac{1}{z-1-1}=\dfrac{1}{z-1}\cdot\dfrac{1}{1-\frac{1}{z-1}}$

$$=\frac{1}{z-1}\cdot\sum_{j=0}^{\infty}\left(\frac{1}{z-1}\right)^j\qquad\text{since }\left|\frac{1}{z-1}\right|<1.$$

$$z^2-2z+3=(z-1)^2+2.$$

So, $$\frac{z^2-2z+3}{z-2}=\left[(z-1)^2+2\right]\cdot\frac{1}{z-1}\left[1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\right]$$

$$=(z-1)\left[1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\right]$$

$$+\frac{2}{z-1}\left[1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\right]$$

$$=z-1+1+\left(\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\right)$$

$$+\left(\frac{2}{z-1}+\frac{2}{(z-1)^2}+\cdots\right)$$

$$= (z-1)+1+\sum_{j=1}^{\infty}\frac{3}{(z-1)^j}. \qquad (|z-1|>1).$$

Example $\quad f(z)=\dfrac{1}{(z-1)(z-2)}$

(a) $|z|<1$. $\quad$ (b) $1<|z|<2$. $\quad$ (c) $|z|>2$.

First, $f(z)=\dfrac{1}{z-2}-\dfrac{1}{z-1}$.

(a) $\dfrac{1}{z-2} = -\dfrac{1}{2}\cdot\dfrac{1}{1-\frac{z}{2}} = -\dfrac{1}{2}\sum_{j=0}^{\infty}\left(\dfrac{z}{2}\right)^j = -\sum_{j=0}^{\infty}\dfrac{z^j}{2^{j+1}}$.

$\dfrac{1}{z-1} = -\dfrac{1}{1-z} = -\sum_{0}^{\infty} z^j$

Hence $f(z)=\sum_{j=0}^{\infty}\left(1-\dfrac{1}{2^{j+1}}\right)z^j = \dfrac{1}{2}+\dfrac{3}{4}z+\dfrac{7}{8}z^2+\cdots$

$(|z|<1)$

(b) $\dfrac{1}{z-2}$: same.

$\dfrac{1}{z-1} = \dfrac{1}{z}\cdot\dfrac{1}{1-\frac{1}{z}} = \dfrac{1}{z}\sum_{j=0}^{\infty}\left(\dfrac{1}{z}\right)^j = \sum_{j=0}^{\infty}\dfrac{1}{z^{j+1}}$

Thus, $f(z) = -\sum_{j=0}^{\infty}\dfrac{1}{z^{j+1}} - \sum_{j=0}^{\infty}\dfrac{z^j}{2^{j+1}} = \cdots -\dfrac{1}{z^2}-\dfrac{1}{z}-\dfrac{z}{4}-\cdots$.

$(1<|z|<2)$

(c) $|z|>2$. $\quad \dfrac{1}{z-1} = \sum_{j=0}^{\infty}\dfrac{1}{z^{j+1}}$ as before.

$\dfrac{1}{z-2} = \dfrac{1}{z}\cdot\dfrac{1}{1-\frac{2}{z}} = \dfrac{1}{z}\sum_{j=0}^{\infty}\left(\dfrac{2}{z}\right)^j = \sum_{j=0}^{\infty}\dfrac{2^j}{z^{j+1}}$.

$f(z) = \sum_{j=0}^{\infty}\dfrac{2^j-1}{z^{j+1}} = \dfrac{1}{z^2}+\dfrac{3}{z^3}+\dfrac{7}{z^4}+\cdots$.

Example $\quad e^{\frac{1}{z}} = 1+\dfrac{1}{z}+\dfrac{1}{2!}\cdot\dfrac{1}{z^2}+\cdots$

Section 3.3 Zeros and Singularities

Terminology.

- $f(z)$ is <u>analytic at z_0</u>, if $f(z)$ is analytic in $|z-z_0|<R$ for some $R>0$.
- z_0 is a <u>zero</u> (point) of $f(z)$, if $f(z_0)=0$.
- z_0 is a <u>zero of order m</u> of $f(z)$, if f is analytic at z_0, and $f(z_0)=f'(z_0)=\cdots=f^{(m-1)}(z_0)=0$ but $f^{(m)}(z_0)\neq 0$.
- $m=1$: <u>simple zero</u> (or simple root).

<u>Theorem</u> $f(z)$ has a zero of order m at z_0

$$\Longleftrightarrow f(z)=(z-z_0)^m g(z)$$

for some $g(z)$ analytic at z_0, $g(z_0)\neq 0$.

<u>Pf</u> By Taylor's series. <u>Q.E.D.</u>

<u>Theorem</u> (Isolation of Zeros) If $f(z)$ is analytic and $f(z_0)=0$, then either $f\equiv 0$ in a neighborhood of z_0 or there is a punched disk about z_0 in which f has no zeros.

<u>Proof</u> $f\equiv 0$ in a n.b.h. of $z_0 \Longleftrightarrow$ all $f^{(n)}(z_0)=0$ $(n=0,1,2,\cdots)$ by Taylor's. If $f\not\equiv 0$ in a n.b.h. of z_0 then $\exists$ (smallest) integer $m\geq 1$ such that $f(z_0)=0, \cdots, f^{(m-1)}(z_0)=0$ but $f^{(m)}(z_0)\neq 0$.
So, the previous Thm $\Longrightarrow f(z)=(z-z_0)^m g(z)$.
$g(z_0)\neq 0 \Longrightarrow g\neq 0$ in a n.b.h. of z_0.
$\Longrightarrow f\neq 0$ in this n.b.h. $\setminus \{z_0\}$. <u>Q.E.D.</u>

Now, consider singularities.

Terminologies.

- If $f(z)$ is analytic in $0<|z-z_0|<R$ for some $R>0$ but not analytic at z_0, then z_0 is an <u>isolated singularity</u> of f
- Let z_0 be an isolated singularity of f. and $f(z)=\sum_{-\infty}^{\infty} a_j (z-z_0)^j$ $(0<|z-z_0|<R)$ (the Laurent series of f at z_0)

— z_0 is a <u>removable</u> singularity if $a_j=0$ for all $j<0$.

— z_0 is <u>a pole of order m</u> (≥ 1) if $a_j=0$ for all $j<-m$, and $a_{-m}\neq 0$

— z_0 is an <u>essential singularity</u> of f, if $a_j\neq 0$ for infinitely many $j<0$.

<u>Examples</u> (a) $f(z)=\begin{cases} \dfrac{\sin z}{z} & z\neq 0 \\ 1 & z=0 \end{cases}$

f is not continuous at $z_0=0$. $z_0=0$ is an isolated singularity of f. But

$f(z)=\dfrac{\sin z}{z} = 1-\dfrac{z^2}{3!}+\dfrac{z^4}{5!}-\cdots \qquad (|z|>0.)$

So, it's a removable singularity.

(b) $g(z)=\dfrac{1}{(z-2)^3}+\sin z$.

$z_0=2$ is a pole of order 3.

(c) $h(z)=e^{\frac{1}{z}}=\sum_{n=0}^{\infty}\dfrac{1}{n!z^n}$

$z_0=0$ is an essential singularity of $h(z)$.

Theorem (a) If f has a removable singularity at z_0, then we can redefine $f(z_0)$ so that the new function is analytic at z_0.

(b) f has a pole of order $m \Longleftrightarrow$

$$f(z) = \frac{g(z)}{(z-z_0)^m}.$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

$\Longleftrightarrow \frac{1}{f(z)}$ has a removable singularity at z_0.

Ideas of proof

(a) $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad 0 < |z-z_0| < R.$$

Redefine $f(z_0) = a_0$.

(b) $$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$(0 < |z-z_0| < R)$$

$$= \frac{1}{(z-z_0)^m}\Big[a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n}\Big] \qquad (a_{-m} \neq 0)$$

$$\equiv \frac{1}{(z-z_0)^m}\, g(z). \qquad g(z_0) \neq 0.$$

$$\frac{1}{f(z)} = \frac{(z-z_0)^m}{g(z)}.$$ Define $\frac{1}{f}$ at z_0 to be 0.

Q.E.D.

Picard's Theorem A function with an essential singularity assumes every complex number, with possibly one exception, as a value in any neighborhood of this singularity.

Example $f(z) = e^{1/z}$, $z_0 = 0$.

Given $c \in \mathbb{C}$. $c \neq 0$. $e^{\frac{1}{z}} = c$ has solutions z close to $z_0 = 0$.

$$\log c = \mathrm{Log}|c| + i \,\mathrm{Arg}\, c + 2k\pi i \quad (k = 0, \pm 1, \pm 2 \cdots).$$

For each k, call this w_k

$$w_k = \mathrm{Log}|c| + i\,\mathrm{Arg}\, c + 2k\pi i.$$

Then, $|w_k| \to \infty$ i.e., $\frac{1}{w_k} \to 0$. Moreover

$$e^{\frac{1}{(1/w_k)}} = e^{w_k} = c \quad (k = 0 \pm 1, \pm 2 \cdots).$$

The possible exception here is 0.

Example $f(z) = \sin\left(1 - \frac{1}{z}\right)$. zeros? singularities?

$f(z) = 0 \iff 1 - z^{-1} = n\pi$, $z = \frac{1}{1 - n\pi}$ $(n \in \mathbb{Z})$

zeros: $z_n = \frac{1}{1 - n\pi}$ $(n \in \mathbb{Z})$. — isolated!

Also, $\frac{d}{dz} \sin\left(1 - \frac{1}{z}\right)\Big|_{z = z_n} = \frac{1}{z^2} \cos\left(1 - \frac{1}{z}\right)\Big|_{z = z_n}$

$$= (1 - n\pi)^2 \cos n\pi \neq 0.$$

So, all zeros $z_n = \frac{1}{1 - n\pi}$ $(n \in \mathbb{Z})$ are simple.

Singularities: only one $z_0 = 0$. This is an essential singularity as $\sin(1 - z^{-1})$ oscillates between ± 1 as $z = \mathrm{Re}\, z \to 0$ (z approaches 0 along the real axis).

Summary {Thm} Let f have an isolated singularity at z_0. Then

(a) z_0 is removable $\iff |f|$ is bounded near $z_0 \iff \lim_{z \to z_0} f(z)$ exists. $\iff f$ can be redefined to be analytic at z_0

(b) z_0 is a pole $\iff \lim_{z \to z_0} |f(z)| = +\infty \iff f(z) = g(z)/(z - z_0)^m$ (for some $m \in \mathbb{N}$) and g analytic at z_0, $g(z_0) \neq 0$

(c) z_0 is essential $\iff |f(z)|$ is neither bounded or goes to ∞ as $z \to z_0 \iff f(z)$ assumes all complex numbers in $\mathbb{C}$ except possibly one value, in any n.b.h. of z_0. Q.E.D.