Section 4 Residue Theory

Section 4.1 The Residue Theorem
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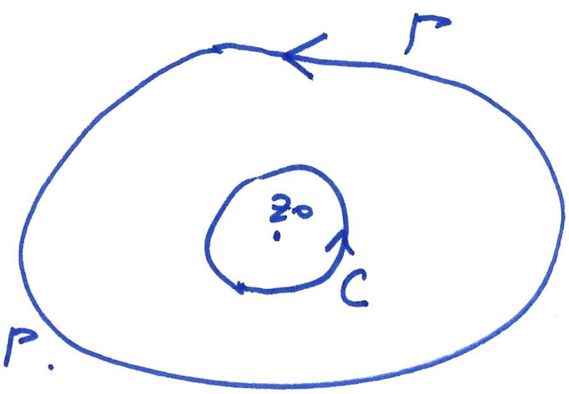
Section 4.1 The Residue Theorem

We consider $\int_\Gamma f(z)dz$.

Assume ① Γ is a simple closed positively oriented contour, surrounding a point z_0.
② $f(z)$ is analytic on Γ and inside Γ except z_0.



The Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

Let $C: |z-z_0| = R$ be inside Γ.

Then $$\oint_\Gamma f(z)dz = \oint_C f(z)dz$$

$$= \sum_{n=-\infty}^{\infty} a_n \oint_C (z-z_0)^n dz = a_{-1}\, 2\pi i$$

So, a_{-1} is important!

<u>Definition</u> If $f(z)$ has an isolated singularity at z_0, then the coefficient a_{-1} of $\frac{1}{z-z_0}$ is the Laurent expansion of f around z_0 is called the residue

of f at z_0 and is denoted

$$\mathrm{Res}(f; z_0) \quad \text{or} \quad \mathrm{Res}(z_0).$$

Note: $a_{-1} = \mathrm{Res}(f; z_0) = \frac{1}{2\pi i} \oint_C f(z)\,dz$

<u>Examples</u> $f(z) = z e^{3/z}$, $z_0 = 0$.

$$f(z) = z\left(1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 \cdot \frac{1}{2!} + \left(\frac{3}{z}\right)^3 \frac{1}{3!} + \cdots\right)$$

$$= z + 3 + \frac{9}{2} \cdot \frac{1}{z} + \frac{3^3}{3!} \frac{1}{z^2} + \frac{3^4}{4!} \cdot \frac{1}{z^3} + \cdots$$

$\mathrm{Res}(f; 0) = \frac{9}{2}$.

Calculate $\oint_{|z|=4} \underbrace{z e^{3/z}}_{f(z)} dz = 2\pi i\, \mathrm{Res}(f; 0) = 9\pi i.$

<u>Theorem</u>

(1) If z_0 is a removable singularity of f, then

$$\mathrm{Res}(f; z_0) = 0.$$

(2) If z_0 is a pole of $f(z)$ of order $m\ (\geq 1)$, then

$$\mathrm{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]$$

<u>Pf</u> (1) After redefining value $f(z_0)$, f is analytic inside and on $\Gamma = C$. So, $a_{-1} = \frac{1}{2\pi i} \int_C f(z)\,dz = 0$.

(2) $$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

$$(z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + a_1(z - z_0)^{m+1} + \cdots$$

$$\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z - z_0)^m f(z)\}$$

$$= \cancel{\frac{1}{(m-1)!}} a_{-1} + a_0 \frac{m!}{(m-1)!}(z - z_0) + \cdots$$

$$a_{-1} = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]. \quad \underline{\text{Q.E.D.}}$$

~~Example~~. Special case $m=1$.

$$\text{Res}(f; z_0) = \lim_{z \to z_0} (z-z_0) f(z).$$

Example

~~For instance~~, $f(z) = \dfrac{p(z)}{Q(z)}$. p, Q: analytic at z_0

z_0: pole of order 1 for $f(z)$. $p(z_0) \neq 0$. Q has a simple zero at z_0.

Then.
$$\text{Res}(f; z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$
$$= \lim_{z \to z_0} (z-z_0) \frac{p(z)}{Q(z)}$$
$$= \lim_{z \to z_0} \frac{p(z)}{\dfrac{Q(z)-Q(z_0)}{z-z_0}} = \frac{p(z_0)}{Q'(z_0)}.$$

Example

$$f(z) = \frac{e^z}{z(z+1)}$$
$$\text{Res}(f; 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{e^z}{z+1} = 1.$$
$$\text{Res}(f; -1) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{e^z}{z} = -e^{-1}$$

Example $f(z) = \cot z = \dfrac{\cos z}{\sin z}$. $z_0 = n\pi$. $(n \in \mathbb{Z})$

simple poles.
$$\text{Res}(f; n\pi) = \frac{\cos z}{(\sin z)'}\bigg|_{z=n\pi} = \frac{\cos n\pi}{\cos n\pi} = 1.$$

Example $f(z) = \dfrac{\cos z}{z^2 (z-\pi)^3}$. $z=0$: pole of order 2

$z=\pi$: pole of order 3

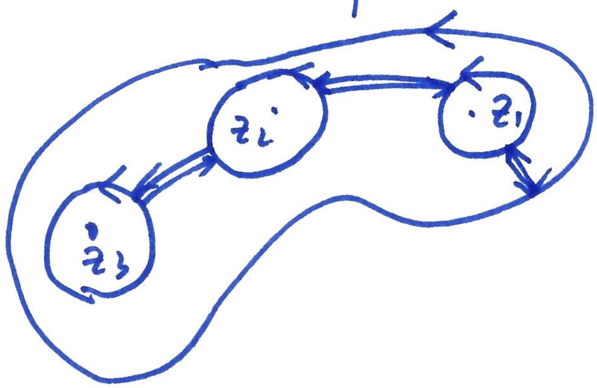
$$\text{Res}(f; 0) = \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z \to 0} \frac{d}{dz}\left[\frac{\cos z}{(z-\pi)^3}\right]$$
$$= \lim_{z \to 0} \frac{-(z-\pi)\sin z - 3\cos z}{(z-\pi)^4} = -\frac{3}{\pi^4}.$$

$$\mathrm{Res}(f;\pi) = \lim_{z\to\pi} \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-\pi)^3 f(z)\right]$$

$$= \lim_{z\to\pi} \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{\cos z}{z}\right]$$

$$= \lim_{z\to\pi} \frac{1}{2}\left[\frac{(6-z^2)\cos z + 4z\sin z}{z^4}\right] = \frac{\pi^2-6}{2\pi^4}.$$

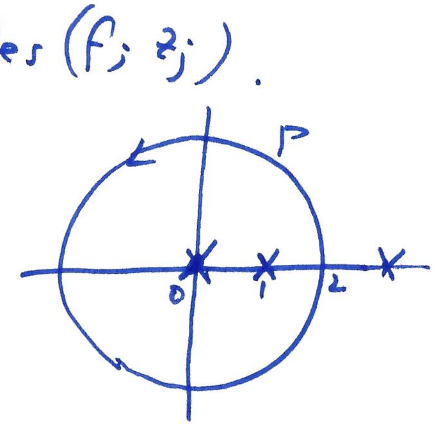
Multiple isolated singularities $z_1, \dots, z_n$.



<u>Cauchy's Residue Theorem</u> If Γ is a simple closed positively oriented contour and f is analytic inside and on Γ except at points $z_1, \dots, z_n$ inside Γ, then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f; z_j).$$

<u>Example</u> $I = \oint_{|z|=2} \frac{1-2z}{z(z-1)(z-3)}\,dz$



$= 2\pi i\,\mathrm{Res}(f;0) + 2\pi i\,\mathrm{Res}(f;1)$

$\mathrm{Res}(f;0) = \lim_{z\to 0} z f(z) = \frac{1}{3}.$

$\mathrm{Res}(f;1) = \lim_{z\to 1} (z-1) f(z) = \frac{1}{2}.$

So, $I = 2\pi i\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5\pi i}{3}.$

Section 4.2 Techniques of Integration

Trigonometric Integrals over $[0, 2\pi]$

$$I = \int_0^{2\pi} U(\cos\theta, \sin\theta)\,d\theta.$$

Let $\boxed{z = e^{i\theta} \quad (0 \le \theta \le 2\pi)}$ $\quad \frac{1}{z} = e^{-i\theta}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}.$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}.$$

$$dz = ie^{i\theta}d\theta = iz\,d\theta. \qquad \boxed{d\theta = \frac{dz}{iz}}$$

$$I = \int_0^{2\pi} U(\cos\theta, \sin\theta)\,d\theta = \oint_{C:|z|=1} F(z)\,dz$$

$$F(z) = U\left[\tfrac{1}{2}\left(z + \tfrac{1}{z}\right), \tfrac{1}{2i}\left(z - \tfrac{1}{z}\right)\right]\frac{1}{iz}.$$

Example 1 $\quad I = \int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\,d\theta = \oint_{|z|=1} \frac{\left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^2}{5 + 4\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]} \cdot \frac{dz}{iz}$

$$= -\frac{1}{4i} \oint_{|z|=1} \frac{(z^2-1)^2}{z^2(2z^2+5z+2)}\,dz$$

$$= -\frac{1}{4i} \oint_{|z|=1} \underbrace{\frac{(z^2-1)^2}{z^2\left(z+\frac{1}{2}\right)(z+2)}}_{g(z)}\,dz = -\frac{1}{4i}\oint_{|z|=1} g(z)\,dz$$

$$= -\frac{1}{4i} \cdot 2\pi i \left[\operatorname{Res}\left(g; -\tfrac{1}{2}\right) + \operatorname{Res}(g; 0)\right]$$

$$\operatorname{Res}\left(g; -\tfrac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \tfrac{1}{2}\right) g(z) = \frac{3}{4}$$

$$\operatorname{Res}(g; 0) = \lim_{z \to 0} \frac{1}{1!} \frac{d}{dz}\left[z^2 g(z)\right] = \lim_{z\to 0} \frac{d}{dz}\left[\frac{(z^2-1)^2}{2z^2+5z+2}\right]$$

$$= -\frac{5}{4}.$$

Example 2 $I = \int_0^{\pi} \frac{d\theta}{2-\cos\theta} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{2-\cos\theta}$

$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2-\cos\theta} = \frac{1}{2}\oint_{|z|=1} \frac{dz/iz}{2-\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]}$$

$$= -\frac{1}{i}\oint_{|z|=1} \frac{dz}{z^2-4z+1}$$

$$z^2-4z+1 = (z-z_1)(z-z_2).$$

$z_1 = 2-\sqrt{3}$ pole, order 1 inside C.

$z_2 = 2+\sqrt{3}$ outside C.

$$I = -\frac{1}{i}\cdot 2\pi i \,\mathrm{Res}(f; z_1)$$

$$f(z) = \frac{1}{z^2-4z+1} = \frac{1}{(z-z_1)(z-z_2)}$$

$$\mathrm{Res}(f; z_1) = \lim_{z\to z_1}(z-z_1)f(z) = \frac{1}{z_1-z_2} = -\frac{1}{2\sqrt{3}}.$$

$$I = -2\pi\cdot\left(-\frac{1}{2\sqrt{3}}\right) = \frac{\pi}{\sqrt{3}}.$$

Some Improper Integrals over $(-\infty, \infty)$

$\int_a^{\infty} f(x)\,dx = \lim_{A\to\infty}\int_a^A f(x)\,dx.$ (if exists)

$\int_{-\infty}^{b} f(x)\,dx = \lim_{B\to-\infty}\int_B^b f(x)\,dx$ (if limit exists)

$$\int_{-\infty}^{\infty} f(x)\,dx = \left(\int_0^{\infty} + \int_{-\infty}^{0}\right) f(x)\,dx$$

Note: $\lim_{A\to\infty}\int_{-A}^{A} x\,dx = 0$ but $\int_{-\infty}^{\infty} x\,dx$ diverges.

$$\text{p.v.}\int_{-\infty}^{\infty} x\,dx = \lim_{A\to+\infty}\int_{-A}^{A} x\,dx = 0$$

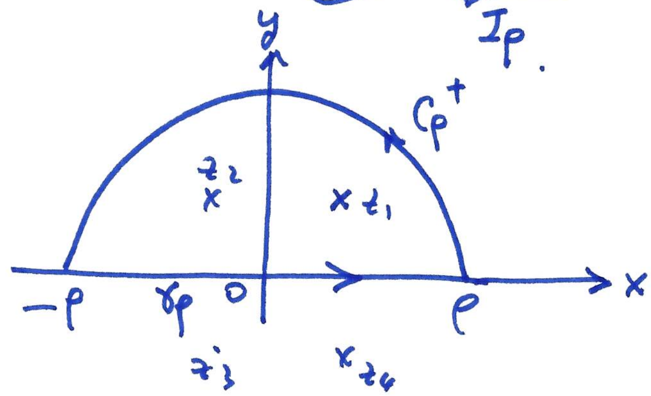
principal value.

$$p.v. \int_{-\infty}^{\infty} f(x)\,dx = \lim_{A \to +\infty} \int_{-A}^{A} f(x)\,dx.$$

If $\int_{-\infty}^{\infty} f(x)\,dx$ converges then $\int_{-\infty}^{\infty} f(x)\,dx = p.v. \int_{-\infty}^{\infty} f(x)\,dx$.

__Example__ $$I = \int_{-\infty}^{\infty} \frac{dx}{x^4+4} = \lim_{\rho \to \infty} \underbrace{\int_{-\rho}^{\rho} \frac{dx}{x^4+4}}_{I_\rho}.$$

$$I_\rho = \int_{\gamma_\rho} \frac{dz}{z^4+4}$$



$$\Gamma_\rho = \gamma_\rho + C_\rho^+$$

$$\int_{\Gamma_\rho} = \int_{\gamma_\rho} + \int_{C_\rho^+}.$$

$$f(z) = \frac{1}{z^4+4} = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

$z_1 = 1+i$, $z_2 = -1+i$, $z_3 = -1-i$, $z_4 = 1-i$

$$\int_{\Gamma_\rho} f(z)\,dz = 2\pi i\left[\operatorname{Res}(f; z_1) + \operatorname{Res}(f; z_2)\right]$$

$$\operatorname{Res}(f; z_1) = \lim_{z \to z_1} (z-z_1) f(z) = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

similar for $\operatorname{Res}(f; z_2)$

$$\boxed{\int_{\Gamma_\rho} f(z)\,dz = \frac{\pi}{4}.}$$

$\lim\limits_{\rho \to +\infty} \int_{C_\rho^+} f(z)\,dz = 0.$ since on C_ρ^+ $|f(z)| \le \frac{1}{|z|^4-4} = \frac{1}{\rho^4-4}$

length $(C_\rho^+) = \pi\rho$.

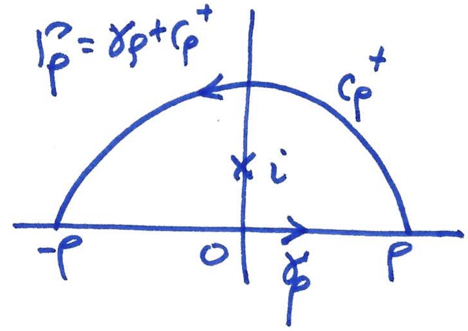
So. $$I = \lim_{\rho \to \infty} \int_{\gamma_\rho} f(z)\,dz = \lim_{\rho \to \infty}\left[\int_{\Gamma_\rho} f(z)\,dz - \int_{C_\rho^+} f(z)\,dz\right] = \frac{\pi}{4}.$$

Example $\quad I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho}\frac{x^2}{(x^2+1)^2}\,dx$

$$= \lim_{\rho\to\infty}\int_{\gamma_\rho} f(z)\,dz \qquad f(z) = \frac{z^2}{(z^2+1)^2} = \frac{z^2}{(z-i)^2(z+i)^2}$$

$$= \lim_{\rho\to\infty}\left[\int_{\Gamma_\rho} f(z)\,dz - \int_{C_\rho^+} f(z)\,dz\right]$$

$\Gamma_\rho = \gamma_\rho + C_\rho^+$



$$= \lim_{\rho\to\infty}\int_{\Gamma_\rho} f(z)\,dz - 0.$$

$$= 2\pi i \operatorname{Res}(f; i)$$

$$= 2\pi i \lim_{z\to i}\frac{1}{1!}\frac{d}{dz}\left[(z-i)^2 f(z)\right] = 2\pi i \lim_{z\to i}\frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right] = 2\pi i\,\frac{1}{4i}$$

$$= \frac{\pi}{2}.$$

Why $\lim_{\rho\to+\infty}\int_{C_\rho^+} f(z)\,dz = 0$?

Reason: On C_ρ^+: $|f(z)| \le \frac{K}{|z|^2} = \frac{K}{\rho^2}$.

length of $C_\rho^+ = \pi\rho$.

So, $\left|\int_{C_\rho^+} f(z)\,dz\right| \le \frac{K}{\rho} \to 0$ as $\rho\to+\infty$.

~~Example I = ∫~~

Example $\quad I = \int_{-\infty}^{\infty}\frac{\cos 3x}{x^2+4}\,dx = \lim_{\rho\to+\infty}\int_{-\rho}^{\rho}\frac{\cos 3x}{x^2+4}\,dx$

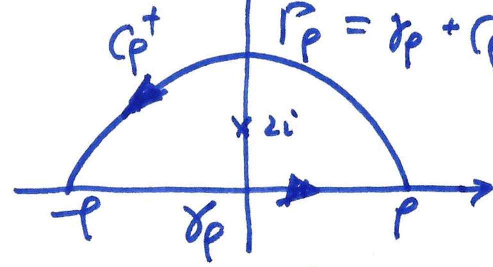
$$= \lim_{\rho\to+\infty}\int_{\gamma_\rho}\frac{\cos 3z}{z^2+4}\,dz$$

Note: $\left|\frac{\cos 3z}{z^2+4}\right| \not\to 0$ if $z\in C_\rho^+$ and $\rho\to+\infty$.

e.g., $z = \pm \rho i$, $\left|\frac{\cos 3z}{z^2+4}\right| = \frac{e^{-3\rho}+e^{3\rho}}{2\,|-\rho^2+4|} \to \infty$ as $\rho \to \infty$.

Try $I = Re(I_0)$, $I_0 = \int_{-\infty}^{\infty} \frac{e^{3ix}}{x^2+4}\,dx$

$f(z) = \frac{e^{3iz}}{z^2+4}$.



$$\int_{\Gamma_\rho} f(z)\,dz = 2\pi i \, Res(f; 2i)$$
$$= 2\pi i \lim_{z\to 2i} (z-2i) f(z)$$
$$= 2\pi i \lim_{z\to 2i} \frac{e^{3iz}}{z+2i} = 2\pi i \cdot \frac{e^{-6}}{4i} = \frac{\pi e^{-6}}{2}$$

$\lim_{\rho\to+\infty} \int_{C_\rho} f(z)\,dz = 0$. since $z \in C_\rho \Rightarrow |f(z)| = |f(x+iy)|$

$$= \frac{|e^{3ix} e^{-3y}|}{|z^2+4|} = \frac{e^{-3y}}{|z^2+4|} \; \cancel{\to 0 \text{ as } y \to +\infty} \; \le \frac{1}{|z^2+4|}.$$

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\pi\rho}{|z^2+4|} \le \frac{\pi\rho}{\rho^2-4} \to 0 \text{ as } \rho \to +\infty.$$

So, $I_0 = \cancel{2}\,\frac{\pi e^{-6}}{2} = \frac{\pi}{2e^6}$. $I = I_0 = \frac{\pi}{2e^6}$.

Jordan's Lemma Let ~~$m>0$,~~ $\deg Q \ge 1 + \deg P$. (P, Q: polynomials). Then

$$\lim_{\rho\to+\infty} \int_{C_\rho^+} e^{imz} \frac{P(z)}{Q(z)}\,dz = 0, \text{ if } m>0.$$

$$\lim_{\rho\to+\infty} \int_{C_\rho^-} e^{imz} \cancel{Q}\,\frac{P(z)}{Q(z)}\,dz = 0, \text{ if } m<0.$$

Skip the proof.

Example $I = \text{p.v.}\int_{-\infty}^{\infty} \frac{\sin x}{x+i}\,dx$

$$= \text{p.v.}\int_{-\infty}^{\infty} \frac{\frac{1}{2i}\left(e^{ix} - e^{-ix}\right)}{x+i}\,dx$$

$$= \underbrace{\text{p.v.}\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{2i(x+i)}}_{I_1} - \underbrace{\text{p.v.}\int_{-\infty}^{\infty} \frac{e^{-ix}\,dx}{2i(x+i)}}_{I_2}$$

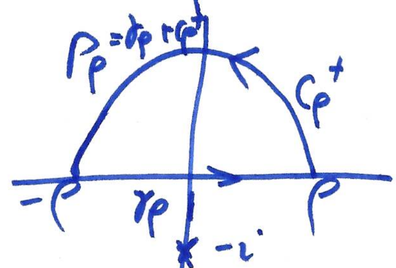
$$= I_1 - I_2.$$

Use Jordan's Lemma: $\lim_{\rho\to+\infty} \int_{C_\rho^+} \frac{e^{iz}}{z+i}\,dz = 0.$

$$I_1 = \lim_{\rho\to+\infty} \int_{\gamma_\rho} \frac{e^{iz}}{2i(z+i)}\,dz$$

$$= \lim_{\rho\to+\infty} \int_{\Gamma_\rho} \frac{e^{iz}}{2i(z+i)}\,dz - \lim_{\rho\to+\infty} \int_{C_\rho^+} \frac{e^{iz}}{z+i}\,dz$$

$$= 0 - 0 = 0.$$



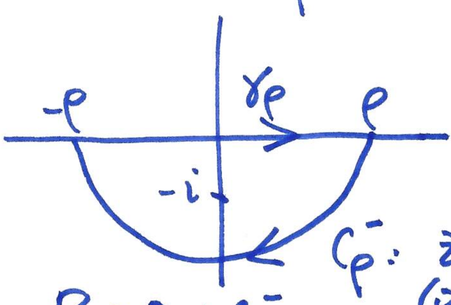
$$\lim_{\rho\to\infty} \int_{C_\rho^-} \frac{e^{-iz}}{z+i}\,dz = 0$$

$$\int_{\Gamma_\rho} \frac{e^{-iz}}{z+i}\,dz = -2\pi i \,\text{Res}(-i)$$

since Γ_ρ is oriented clockwise

$$= -2\pi i \lim_{z\to -i} e^{-iz}$$

$$= \frac{-2\pi i}{e}$$



C_ρ^-: $z = \rho e^{-it}$ $(0 \le t \le \pi)$

$\Gamma_\rho = \gamma_\rho + C_\rho^-$

So, $I_2 = \frac{1}{2i} \lim_{\rho\to\infty} \int_{\gamma_\rho} = \frac{1}{2i}\lim_{\rho\to\infty}\int_{\Gamma_\rho} - \frac{1}{2i}\underbrace{\int_{C_\rho^-}}_{=0}$

$$= \frac{1}{2i}\cdot\left(-\frac{2\pi i}{e}\right) = -\frac{\pi}{e}.$$

$$I = I_1 - I_2 = 0 - \left(-\frac{\pi}{e}\right) = \frac{\pi}{e}.$$

Indented Contours

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_a^{b-\varepsilon} f(x)\,dx \qquad \text{(if limit exists)}$$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^{b} f(x)\,dx. \qquad \text{(if limit exists)}$$

__Example__ $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\varepsilon\to 0^+} 2\sqrt{x}\Big|_{x=\varepsilon}^{x=1}$

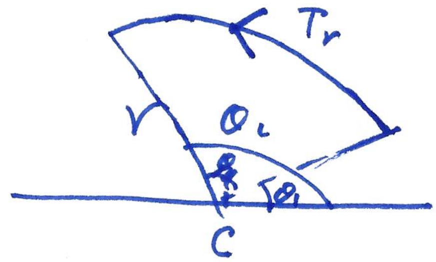
$= 2.$

$$\text{p.v.} \int_a^b f(x)\,dx = \lim_{r\to 0^+}\int_{a+r}^{b-r} f(x)\,dx.$$

__Lemma__ If $f(z)$ has a simple pole at $z=c$ and T_r is: $z = c+re^{i\theta}$ $(\theta_1\le\theta\le\theta_2)$. Then

$$\lim_{r\to 0^+}\int_{T_r} f(z)\,dz = i(\theta_2-\theta_1)\,\mathrm{Res}(f;c).$$

Special case. $\theta_1=0$, $\theta_2=\pi$.



__Pf.__ $f(z) = \dfrac{a_{-1}}{z-c} + \underbrace{\sum_{n=0}^{\infty} a_n(z-c)^n}_{g(z):\ \text{analytic at } c}$.

$|g(z)|\le M$: bounded.

$$\left|\int_{T_r} g(z)\,dz\right| \le M\,\text{length}(T_r) \to 0 \text{ as } r\to 0.$$

$$\int_{T_r} f(z)\,dz = \int_{T_r}\frac{a_{-1}}{z-c}\,dz = a_{-1}\int_{\theta_1}^{\theta_2}\frac{1}{re^{i\theta}}\,rie^{i\theta}\,d\theta$$

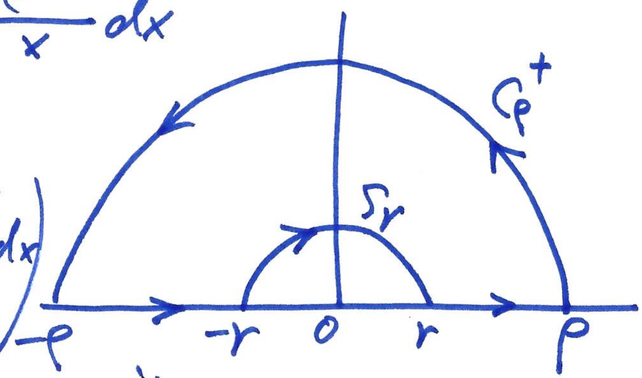
$$= i(\theta_2-\theta_1)\,a_{-1} = i(\theta_2-\theta_1)\,\mathrm{Res}(f;c).$$

__Q.E.D.__

Example $I = p.v.\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$

$f(z) = \frac{e^{iz}}{z}$.

$$I = \lim_{\substack{\rho\to\infty \\ r\to 0^+}} \left(\int_{-\rho}^{-r}\frac{e^{ix}}{x}\,dx + \int_{r}^{\rho}\frac{e^{ix}}{x}\,dx\right)$$



$$\left(\int_{-\rho}^{-r} + \int_{S_r} + \int_{r}^{\rho} + \int_{C_\rho^+}\right)\frac{e^{iz}}{z}\,dz = 0.$$

Hence,
$$\int_{-\rho}^{-r}\frac{e^{ix}}{x}\,dx + \int_{r}^{\rho}\frac{e^{ix}}{x}\,dx = -\int_{S_r}\frac{e^{iz}}{z}\,dz - \underbrace{\int_{C_\rho^+}\frac{e^{iz}}{z}\,dz}_{\to 0 \text{ Jordan's Lemma}}$$

Lemma $\Rightarrow \lim_{r\to 0^+}\int_{S_r}\frac{e^{iz}}{z}\,dz$

$= -\pi \operatorname{Res}(0) = -i\pi.$

↑ note the orientation of S_r.

$$I = p.v.\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx = -(-i\pi) = i\pi.$$

Example $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2}\,p.v.\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$

$= \frac{1}{2}\pi.$

Example $p.v.\int_{-\infty}^{\infty}\frac{x e^{2ix}}{x^2-1}\,dx = i\pi\cos 2.$