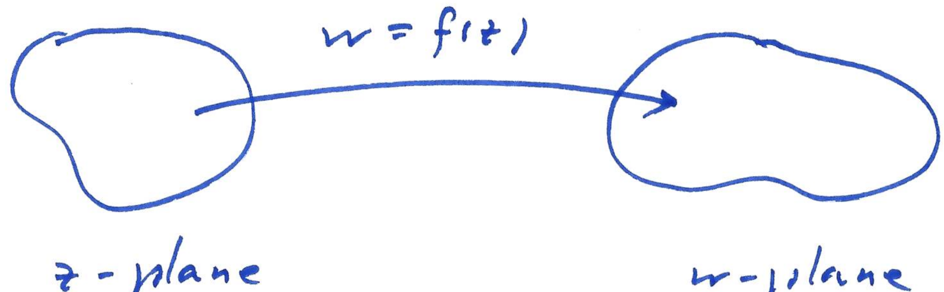
Section 5 Conformal Mapping

Section 5.1 Concept of Conformal Mapping. Riemann Mapping Theorem

We first consider the invariance of harmonic functions in 2 dimensional space.

i.e., $\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.

Assume $\phi = \phi(u,v)$ is harmonic in a nice domain in the w-plane



Suppose $w = f(z)$ is analytic.
Write it as $w = u(x,y) + i\,v(x,y)$.

<u>Claim</u> $\phi(u(x,y), v(x,y))$ is still harmonic.

One way to prove it is to verify it directly using the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x.$$

Or: Let $\psi = \psi(u,v)$ be a conjugate harmonic function to $\phi(u,v)$. Then
$g(u,v) = \phi(u,v) + i\,\psi(u,v)$ is analytic in (u,v)

So, we have the C-R. equations

$$\phi_u = \psi_v, \quad \phi_v = -\psi_u.$$

Now,

$$\frac{\partial}{\partial x}\left(\phi(u(x,y), v(x,y))\right)$$

$$= \phi_u u_x + \phi_v v_x$$

$$= \cancel{\phi_u}\, \psi_v u_x - \psi_u v_x$$

$$\frac{\partial}{\partial y}\left(\psi(u(x,y), v(x,y))\right)$$

$$= \psi_u u_y + \psi_v v_y$$

$$= \psi_u \cancel{v_y} (-v_x) + \psi_v u_x$$

So, $\frac{\partial}{\partial x}(\phi(\dots)) = \frac{\partial}{\partial y}(\psi(\dots))$

Same: $\frac{\partial}{\partial y}(\phi(\dots)) = -\frac{\partial}{\partial x}(\psi(\dots))$.

Another way to look at this

$\phi = \phi(x,y)$: harmonic.

$w = f(z)$: 1-1 (i.e. $z_1 \neq z_2 \Longrightarrow f(z_1) \neq f(z_2)$)

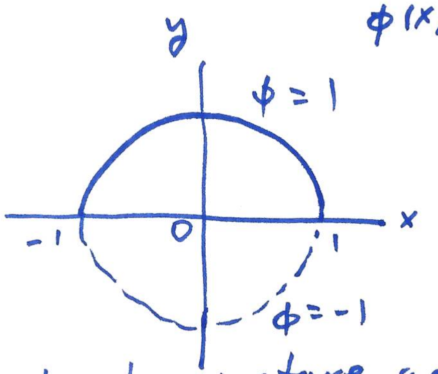
$\Longrightarrow z = f^{-1}(w) = x(u,v) + i\,y(u,v)$.

Then $\phi(x(u,v), y(u,v))$ is harmonic in (u,v).

<u>Example</u> Find $\phi(x,y)$ harmonic in $|z|<1$ and

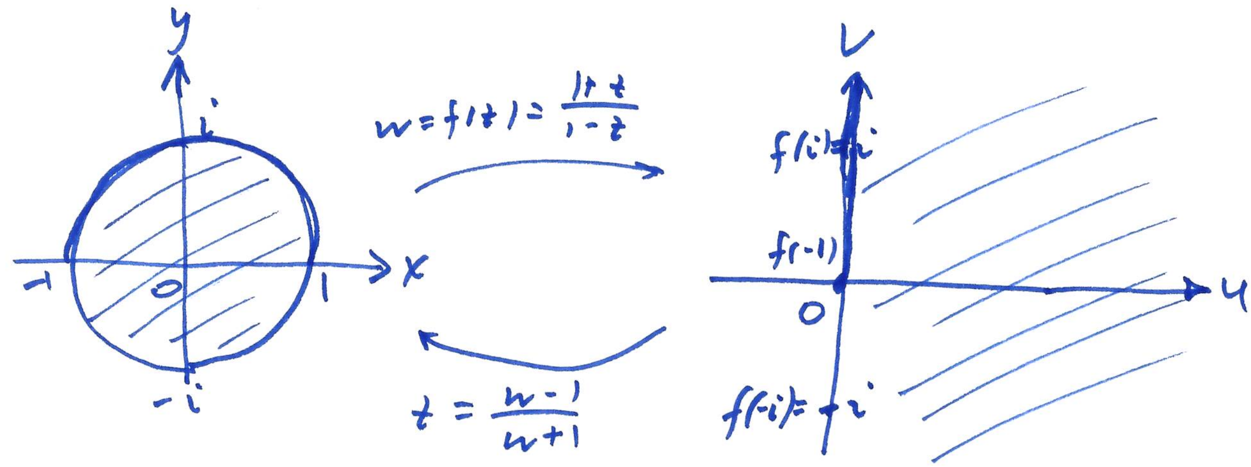
$\phi(x,y) \to 1$ on upper half-circle

$\phi(x,y) \to -1$ on lower half-circle



Solution $w = f(z) = \frac{1+z}{1-z}$

ϕ = temperature, e.g.



Now find $\psi=\psi(u,v)$ harmonic in the right-half plane such that

$$\psi(u,v) \to +1 \quad \text{on positive } v\text{-axis}$$
$$ \to -1 \quad \text{on negative } v\text{-axis}$$

Solution: $\psi(u,v)=\frac{2}{\pi}\operatorname{Arg} w$

$$\phi(x,y)=\psi(u(x,y),v(x,y))=\frac{2}{\pi}\operatorname{Arg}(f(z))=\frac{2}{\pi}\operatorname{Arg}\left(\frac{1+z}{1-z}\right)$$
$$=\frac{2}{\pi}\tan^{-1}\frac{2y}{1-x^2-y^2}.$$

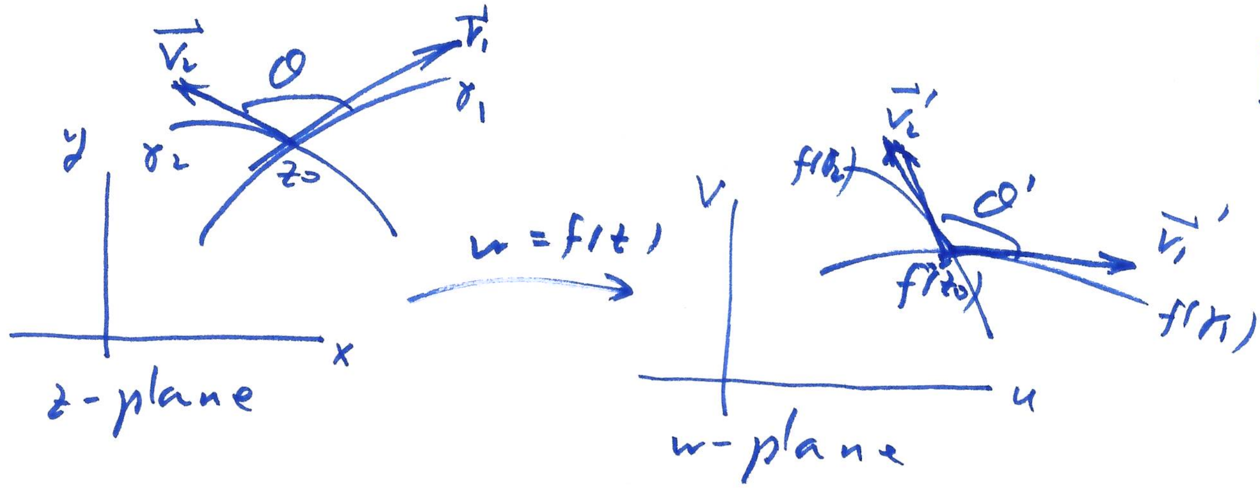
Some geometrical considerations

Definition $f(z)$ is conformal at z_0, if it preserves that angle at z_0. i.e., for any two curves γ_1, γ_2 passing through the point z_0, the angle

$$\langle \gamma_1, \gamma_2 \rangle_{z=z_0} = \langle f(\gamma_1), f(\gamma_2) \rangle_{w=f(z_0)}.$$

The angle $\langle \gamma_1, \gamma_2 \rangle_{z=z_0}$ is defined to be the angle between the tangent of γ_1 at z_0 and that of γ_2 at z_0.

If $\gamma_j: z_j=z_j(t)$ $z_0=z_1(t_0)=z_2(t_0)$, then the angle is $\langle z_1'(t_0), z_2'(t_0) \rangle$.



$$\theta = \theta'$$

If $f(z)$ is conformal at every point in a domain D, then $f(z)$ is conformal in D.

<u>Theorem</u> An ~~all~~ analytic function f is conformal at every point z_0 for which $f'(z_0) \neq 0$.

<u>Proof</u>. Consider γ: $z = z(t)$ with $z_0 = z(t_0)$.

$f(\gamma)$: $w = w(t) = f(z(t))$.

$w_0 = w(t_0) = f(z(t_0)) = f(z_0)$.

The chain rule: $w'(t_0) = f'(z_0)\, z'(t_0)$

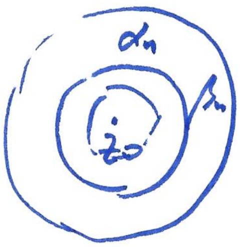
$$\arg w'(t_0) = \underset{(f'(z_0) \neq 0)}{\arg f'(z_0)} + \arg z'(t_0)$$

The change of angle is $\arg f'(z_0)$, fixed. (indep. of the curve γ.) <u>Q.E.D.</u>

Also, we have

<u>Thm</u> If f is analytic in D and ~~$f'(z_0) \neq 0$ in D~~ $f'(z_0) \neq 0$ for some $z_0 \in D$, then f is 1-1 in a neighborhood of z_0.

<u>Proof</u>

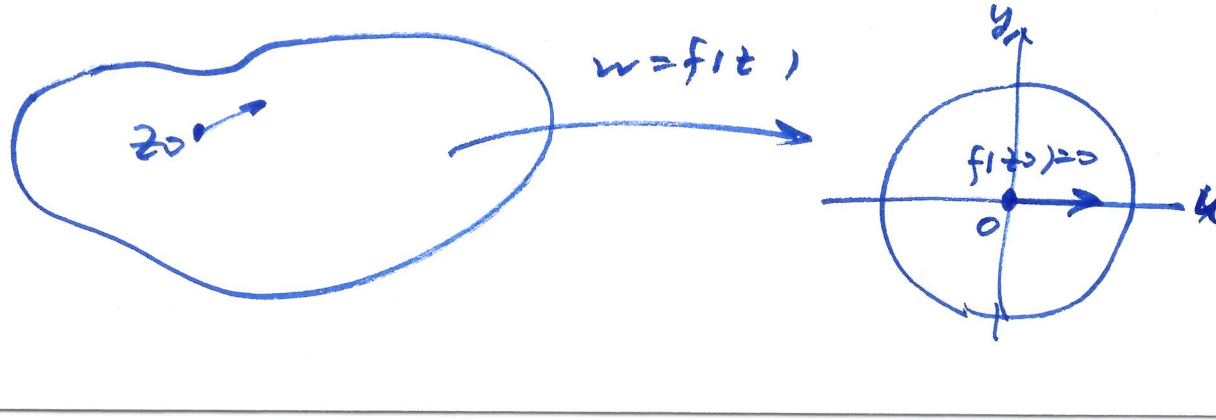


Otherwise $\exists\ \alpha_n, \beta_n$:

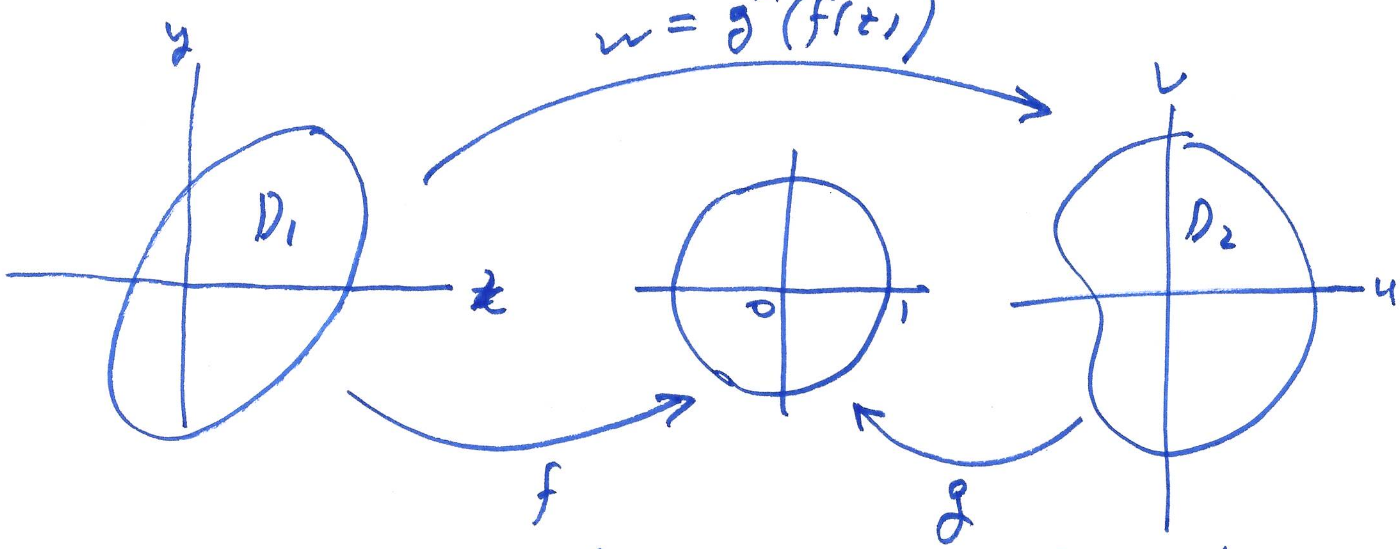
- $\alpha_n \neq \beta_n \quad (n = 1, 2, \cdots)$
- $\alpha_n \to z_0$
  $\beta_n \to z_0$
- $f(\alpha_n) = f(\beta_n) \quad (n = 1, 2, \cdots)$

Then
$$0 = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = \frac{1}{\beta_n - \alpha_n}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\beta_n}dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\alpha_n}dz\right]$$
$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-\alpha_n)(z-\beta_n)}dz$$
$$\longrightarrow \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$
$$= f'(z_0),$$ a contradiction. <u>Q.E.D.</u>

<u>Riemann Mapping Theorem</u> Let $D \subsetneq \mathbb{C}$ be a simply connected domain ($\neq \mathbb{C}$). Then $\exists$ 1-1, analytic function that maps D to $|z| < 1$. Moreover, for any $z_0 \in D$ and any direction from z_0, one can choose that function so that it maps z_0 to 0 and that direction to that of the positive real ~~x~~ axis. ~~direction~~ Such restrictions lead to the uniqueness



In the next two subsections, we construct analytic mappings.



Mapping of simply connected domains.

One-way is enough, since we can use the inverse function for the other.

Section 5.2 Möbius Transformations

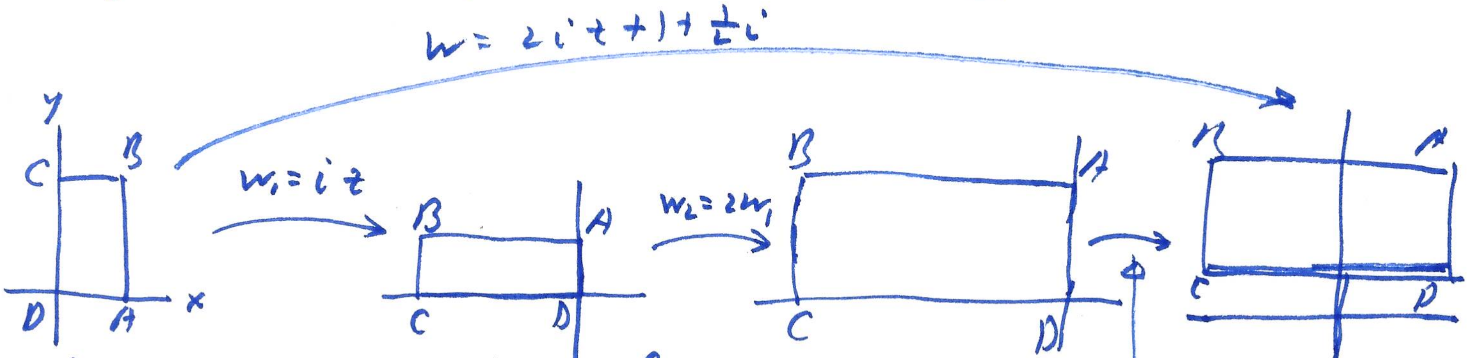
Defined by
$$w = f(z) = \frac{az+b}{cz+d} \qquad (ad \neq bc).$$

Translation $\quad w = f(z) = z + c. \qquad (c \in \mathbb{C})$

Rotation $\quad w = f(z) = e^{i\phi} z. \qquad (\phi \in \mathbb{R})$

Affine transformation
$$w = f(z) = az + b \qquad (a, b \in \mathbb{C})$$

$$\left[\ a = |a| e^{i\phi}, \quad az = |a| e^{i\phi} z.\ \right]$$



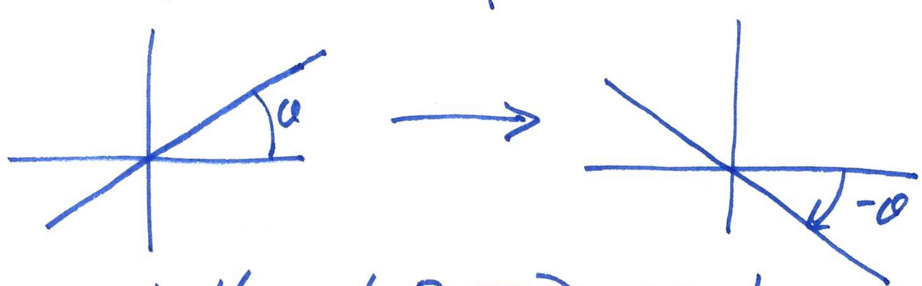
Dilation $\quad z \to \lambda z. \quad \lambda$: real.

$w_3 = w_2 + 1 + \frac{1}{2}i.$

Inversion $\quad w = f(z) = \frac{1}{z}$

(a) lines through 0 $\Longrightarrow$ lines through 0

$$z = \rho e^{i\alpha} \Longrightarrow w = \frac{1}{\rho} e^{-i\alpha}$$



(b) lines not through 0 $\Longrightarrow$ circles.

$\frac{1}{\infty} = 0 \qquad \infty \to 0.$

$Ax+By=C \quad (C\neq 0) \quad (A, B, C \in \mathbb{R})$

$w=\frac{1}{z} \quad z=\frac{1}{w}=\frac{\bar{w}}{|w|^2}=\frac{u-iv}{u^2+v^2} \quad (w=u+iv).$

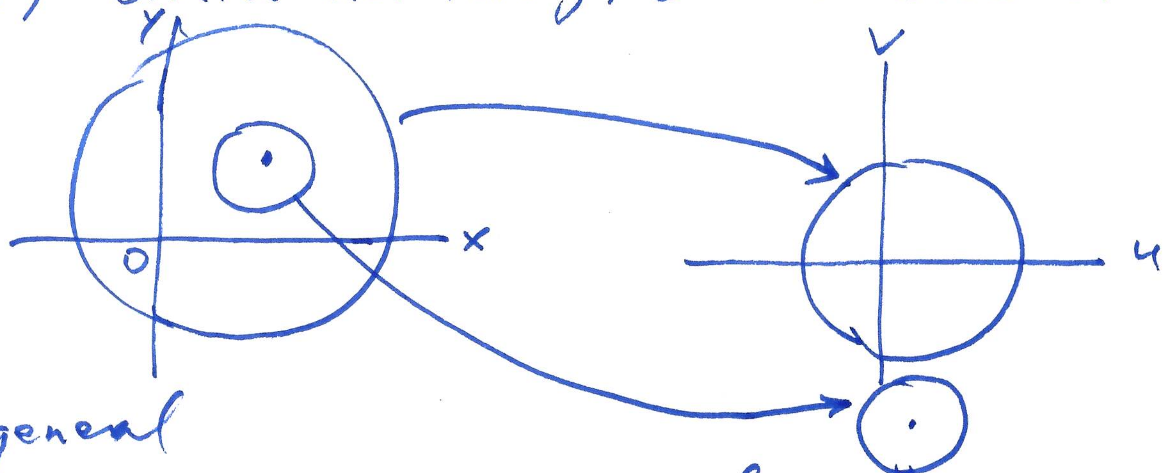
$z=x+iy$

$x=\frac{u}{u^2+v^2}, \quad y=-\frac{v}{u^2+v^2}.$

$Ax+By=C \Longrightarrow u^2+v^2-\frac{A}{C}u+\frac{B}{C}v=0.$

(c) Circles passing through $O \Longrightarrow$ lines (opposite to (b)).

(d) circles not through $O \Longrightarrow$ circles.



In general

$$\frac{az+b}{cz+d}=\frac{a}{c}+\frac{b-\frac{ad}{c}}{cz+d}$$

$w_1=cz+d, \quad w_2=\frac{1}{w_1}, \quad w=\left(b-\frac{ad}{c}\right)w_2+\frac{a}{c}.$

Theorem Let f be a Möbius transformation. Then,

① f is the composition of finite sequence of translations, ~~...~~ dilations, rotations, inversions.

② f maps $\mathbb{C}\cup\{\infty\}$ to $\mathbb{C}\cup\{\infty\}$, 1-1, onto.

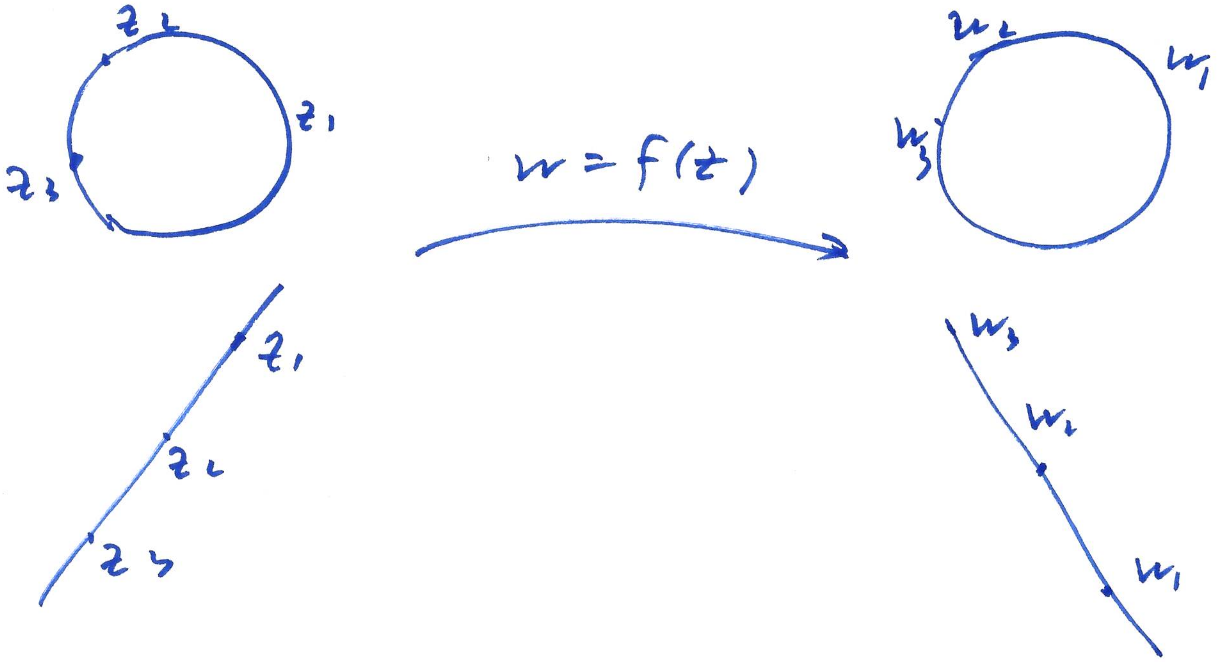
③ f maps circles and lines to circles and lines

④ f is conformal at every point except its pole.

Theorem. f, g Möbius $\Longrightarrow$ $f\circ g$ is Möbius, too.

The cross-ratio technique.

Given a circle or line ℓ_z in z-plane and 3 distinct points z_1, z_2, z_3 on this circle or line. (one of them can be ∞ if this is a line).



Similarly, given 3 points w_1, w_2, w_3 on a line or circle ℓ_w in the w-plane.

Then, the unique Möbius transformation $w=f(z)$ that maps ℓ_z to ℓ_w and $f(z_j)=w_j$ $(j=1,2,3)$ is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3).$$

$$\boxed{\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}}$$

(z, z_1, z_2, z_3)

$$\frac{z \to z_1 \to z_2 \to z_3}{z \to z_3 \to z_2 \to z_1}$$

Example Find a Möbius transformation:

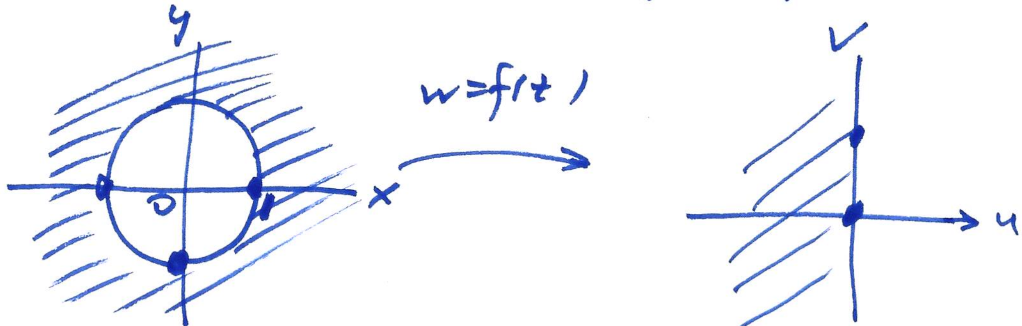
$$0 \to i, \quad 1 \to 2, \quad -1 \to 4$$

Solution $(w, i, 2, 4) = (z, 0, 1, -1) = \dfrac{2z}{z+1}$.

$$\frac{(w-i)(2-4)}{(w-4)(2-i)} = \frac{-2(w-i)}{(w-4)(2-i)}$$

$$w = \frac{(16-6i)z + 2i}{(6-2i)z+2} = \frac{(8-3i)z+i}{(3-i)z+1}.$$

Example Find a Möbius transformation that maps $D_1: |z|>1$ onto $D_2: \operatorname{Re} w < 0$



$z_1 = 1, \quad z_2 = -i, \quad z_3 = -1$

$w_1 = 0, \quad w_2 = i \quad w_3 = \infty$

orientation!

$$(w, w_1, w_2, w_3) = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{w-w_1}{w_2-w_1}$$

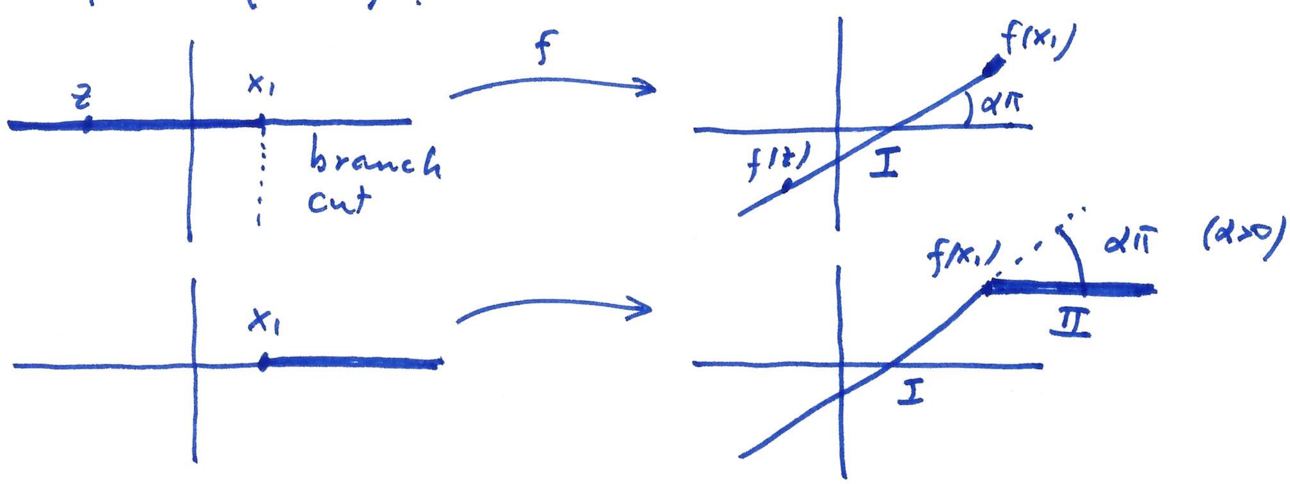
$$= \frac{w}{i}$$

$$(z, z_1, z_2, z_3) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(z-1)(1-i)}{(z+1)(-i-1)}$$

$$\frac{w}{i} = \frac{(z-1)(1-i)}{(z+1)[-(1+i)]} \Longrightarrow \boxed{w = \frac{1-z}{1+z}}$$

Section 5.3 The Schwarz-Christoffel Transformation

Example Let $x_1 \in \mathbb{R}$, $-1 < \alpha < 1$. Consider $f(z)$ such that $f'(z) = (z - x_1)^{\alpha}$.



$$f: (-\infty, x_1) \longrightarrow \mathrm{I}.$$
$$(x_1, \infty) \longrightarrow \mathrm{II}.$$

If $z \in (-\infty, x_1)$: $\arg f'(z) = \arg (z - x_1)^{\alpha} = \alpha \arg (z - x_1) = \alpha\pi$.

If $z \in (x_1, \infty)$: $\arg f'(z) = \alpha \arg (z - x_1) = \alpha \cdot 0 = 0$

[Neglect multiples of 2π.]

Recall: $g(z)$ analytic. $g'(z_0) \neq 0$

Then, $\forall \gamma$: $z = z(t)$, $z(t_0) = z_0$. Then for $w(t) = g(z(t))$:

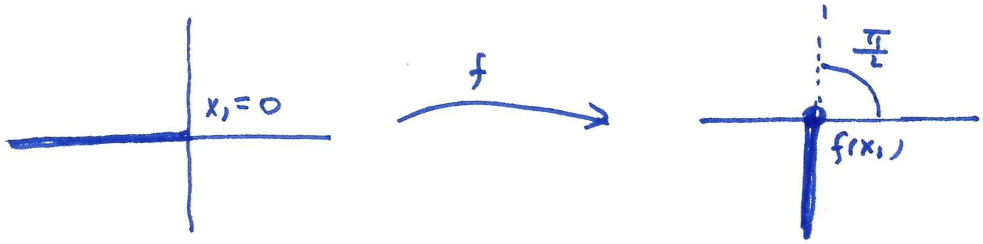
$w'(t_0) = g'(z_0)\, z'(t_0)$

$\arg w'(t_0) = \arg g'(z_0) + \arg z'(t_0)$

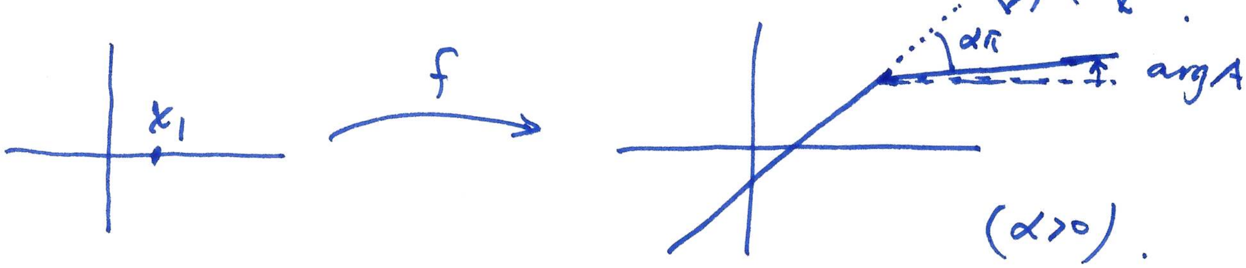
Every curve through z_0 is rotated through the same angle $\arg g'(z_0)$.

Back to $f'(z) = (z - x_1)^{\alpha}$. $[z, x_1]$ is a curve. every point on this curve has tangent // x-axis. The image of $[z, x_1]$ under f is also a curve, all of whose tangents make an angle $\alpha\pi$ with the real axis. So, a line.

Example $\quad f(z) = \frac{2}{3} z^{3/2}, \quad f'(z) = z^{1/2}.$



Example $\quad$ Suppose $f'(z) = A(z - x_1)^{\alpha}$. $\quad \begin{pmatrix} x_1 \in \mathbb{R}, \ -1 < \alpha < 1, \\ A \in \mathbb{C} \end{pmatrix}.$



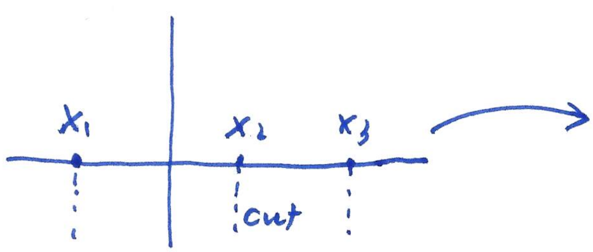
Schwarz-Christoffel transformations

$$f'(z) = A(z - x_1)^{\alpha_1} \cdots (z - x_n)^{\alpha_n}$$

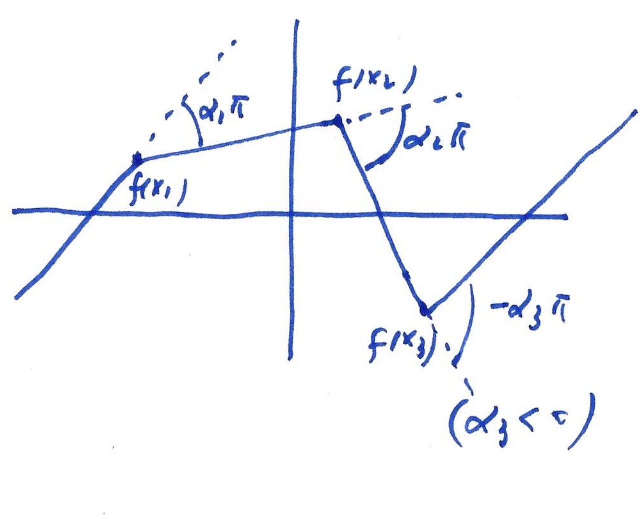
$A \in \mathbb{C}, \ -\infty < x_1 < x_2 < \cdots < x_n < \infty$.

all $\alpha_j \in (-1, 1)$, $\quad j = 1, \cdots, n$.

$$f(z) = A \int_0^z (\zeta - x_1)^{\alpha_1} (\zeta - x_2)^{\alpha_2} \cdots (\zeta - x_n)^{\alpha_n} d\zeta + B.$$



$\arg(z - x_j) \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$



$$\arg f'(z) = \arg A + \alpha_1 \arg(z - x_1) + \cdots + \alpha_n \arg(z - x_n).$$

Interval	Angle of image
$(-\infty, x_1)$	$\arg A + \alpha_1\pi + \alpha_2\pi + \cdots + \alpha_n\pi$
(x_1, x_2)	$\arg A + \alpha_2\pi + \cdots + \alpha_n\pi$
$\vdots$	
(x_{n-1}, x_n)	$\arg A + \alpha_n\pi$
(x_n, ∞)	$\arg A$

<u>Theorem</u> Let P be a positively oriented polygon having consecutive corners at $w_1, w_2, \cdots, w_n$ with corresponding right turn angles θ_j $(j = 1, 2, \cdots, n)$. Then there exists a one-to-one conformal map from the upper half-plane onto the interior of P. Moreover, this map can be constructed as the following Schwarz-Christoffel transformation:

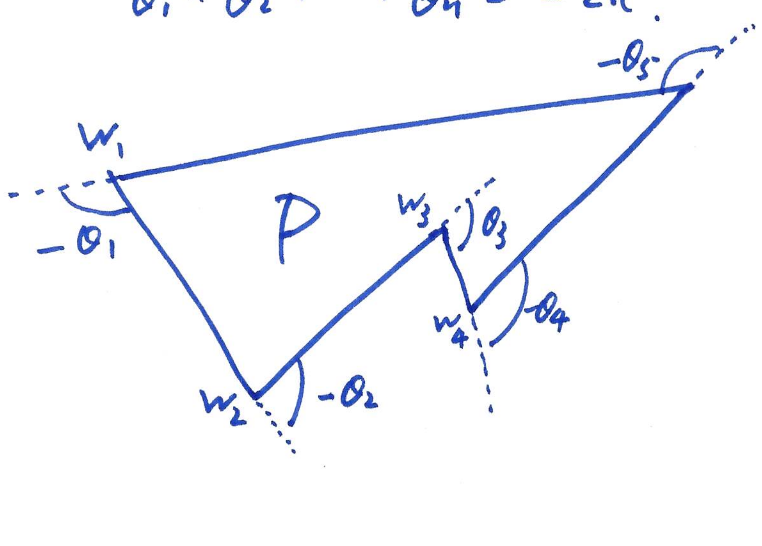
$$f(z) = A \int_0^z (\zeta - x_1)^{\theta_1/\pi} \cdots (\zeta - x_{n-1})^{\theta_{n-1}/\pi} \, d\zeta + B$$

with $-\infty < x_1 < x_2 < \cdots < x_{n-1} < \infty$, and

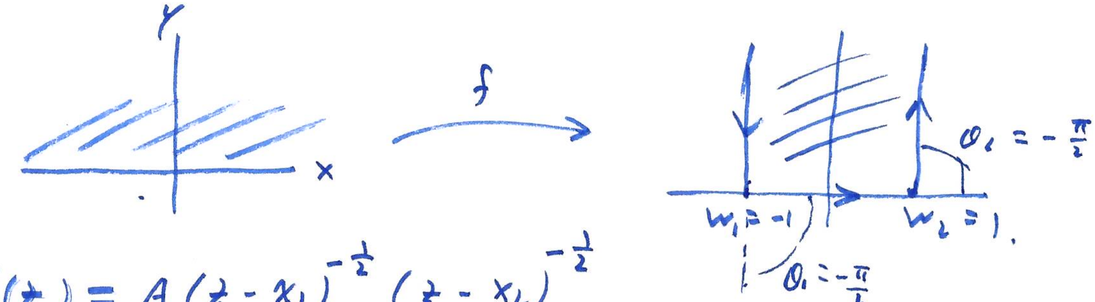
$$f(x_1) = w_1, \quad f(x_2) = w_2, \quad \cdots, \quad f(x_{n-1}) = w_{n-1}, \quad f(\infty) = w_n.$$

Note: $\theta_j \in (-\pi, \pi)$, $j = 1, 2, \cdots, n$.

$\theta_1 + \theta_2 + \cdots + \theta_n = -2\pi$.



Example Determine a Schwarz-Christoffel transformation which maps the upper half-plane onto the semi-infinite strip $|\mathrm{Re}\, w| < 1$, $\mathrm{Im}\, w > 0$.



$$f'(z) = A(z - x_1)^{-\frac{1}{2}}(z - x_2)^{-\frac{1}{2}}$$

Let $x_1 = -1$, $x_2 = 1$.

$$f(z) = A\int_0^z (s+1)^{-1/2}(s-1)^{-1/2}\,ds + B$$

$$= \frac{A}{i}\int_0^z \frac{ds}{\sqrt{1-s^2}} + B$$

$$= \frac{A}{i}\sin^{-1} z + B$$

Setting $f(-1) = w_1 = -1$: $\quad -iA\sin^{-1}(-1) + B = -1$

setting $f(1) = w_2 = 1$: $\quad -iA\sin^{-1}(1) + B = 1$

$\Longrightarrow$ $B = 0$, $A = 2i/\pi$.

$$f(z) = \frac{2}{\pi}\sin^{-1} z.$$