Section 5 Conformal Mapping

Section 5.1 Concept of Conformal Mapping

Riemann Mapping Theorem

We first consider the invariance of harmonic functions in 2-dimensional space.

\[ \Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \]

Assume \( \phi = \phi(u,v) \) is harmonic in a nice domain in the \( w \)-plane.

Suppose \( w = f(z) \) is analytic. Write it as \( w = u(x,y) + iv(x,y) \).

Claim \( \phi(u(x,y), v(x,y)) \) is still harmonic.

One way to prove it is to verify it directly using the Cauchy-Riemann equations:

\[ U_x = V_y, \quad U_y = -V_x. \]

Or: Let \( \psi = \psi(u,v) \) be a conjugate harmonic function to \( \phi(u,v) \). Then

\[ g(u,v) = \phi(u,v) + i \psi(u,v) \] is analytic in \( (u,v) \).

So, we have the C-R equations:

\[ \phi_u = \psi_v, \quad \phi_v = -\psi_u. \]
\[ \frac{\partial}{\partial x} \left( \phi(x,y), v(x,y) \right) \]
\[ = \phi_u u_x + \phi_v v_x \]
\[ = \phi_u x \phi_y - \phi_u y \phi_x \]
\[ \frac{\partial^2}{\partial y} \left( \phi(x,y), v(x,y) \right) \]
\[ = \phi_u u_y + \phi_v v_y \]
\[ = \phi_u y \phi_x - \phi_u x \phi_y \]
So,
\[ \frac{\partial}{\partial x} (\phi_{xx}) = \frac{\partial}{\partial y} (\phi_{yy}) \]
Same:
\[ \frac{\partial}{\partial y} (\phi_{yy}) = -\frac{\partial}{\partial x} (\phi_{xx}) \]

Another way to look at this:
\[ \phi = \phi(x,y) \text{ harmonic.} \]
\[ w = f(\phi) : \text{ 1-1 } \quad (\text{i.e. } z \neq z' \Rightarrow f(z) \neq f(z')) \]
\[ \Rightarrow \quad z = f^{-1}(w) = \kappa(u,v) + i' y/(u,v) \]
Then
\[ \phi(x(u,v), y(u,v)) \text{ is harmonic in } (u,v) . \]

Example
Find \( \phi(x,y) \) harmonic in \( |z| < 1 \) and
\[ \phi(x,y) \to 1 \text{ on upper half-circle} \]
\[ \phi(x,y) \to -1 \text{ on lower half-circle} \]

Solution
\[ w = f(\phi) = \frac{z + \phi}{1 - \phi} \]
\[ \phi = \text{ temperature, e.g.} \]
Now find \( y = y(u, v) \) harmonic in the right-half plane such that:

- \( y(u,v) \to +1 \) on positive \( v \)-axis
- \( y(u,v) \to -1 \) on negative \( v \)-axis

Solution:

\[
y(u,v) = \frac{1}{u} \text{Arg} w
\]

\[
\phi(x,y) = y(u(x,y), v(x,y)) = \frac{1}{u} \text{Arg} (1/z) = \frac{2}{\pi} \text{Arg} \left( \frac{1 + iz}{1 - iz} \right)
\]

\[
= \frac{2}{\pi} \tan^{-1} \frac{2y}{1-x^2-y^2}.
\]

Some geometrical considerations

Definition: \( f(z) \) is conformal at \( z_0 \) if it preserves the angle at \( z_0 \), i.e., for any two curves \( \xi_1, \xi_2 \) passing through the point \( z_0 \), the angle

\[
\angle \xi_1, \xi_2 \to z_0 = \angle f(\xi_1), f(\xi_2) \to w = f(z_0).
\]

The angle \( \angle \xi_1, \xi_2 \to z_0 \) is defined to be the angle between the tangents of \( \xi_1 \) at \( z_0 \) and that of \( \xi_2 \) at \( z_0 \).

If \( \xi_1, z_1 = z_1(t) \to z_0 = z_1(\alpha) = z_1(\beta) \), then the angle is

\[
\langle z_1'(\alpha), z_1'(\beta) \rangle.
\]
If \( f(z) \) is conformal at every point in a domain \( D \), then \( f(z) \) is conformal in \( D \).

**Theorem** An analytic function \( f \) is conformal at every point \( z_0 \) for which \( f'(z_0) \neq 0 \).

**Proof.** Consider \( y : z = t(z_0) \) with \( z_0 = z(t_0) \).

\[ f(y) : w = w(t) = f(t(z_0)) \]
\[ w_0 = w(t_0) = f(t(z_0)) = f(z_0)_0 \]

The chain rule:

\[ w'(t_0) = f'(t(z_0)) \cdot t'(t_0) \]
\[ \arg w'(t_0) = \arg f'(t(z_0)) + \arg t'(t_0) \]

The change of angle is \( \arg f'(z_0) \), fixed.

(Independence of the curve \( y \)). \( \Box \)

Also, we have

*If \( f \) is analytic in \( D \) and \( f'(z_0) \neq 0 \) for some \( z_0 \in D \), then \( f \) is 1-1 in a neighborhood of \( z_0 \).*
Proof

\[ \begin{align*}
\text{ Otherwise } \exists \alpha_n, \beta_n: \\
&\cdot \alpha_n \neq \beta_n \quad (n = 1, 2, \ldots) \\
&\cdot \alpha_n \to \infty \\
&\cdot \beta_n \to \infty \\
&\cdot f(\alpha_n) = f(\beta_n) \quad (n = 1, 2, \ldots)
\end{align*} \]

Then

\[ \begin{align*}
0 &= \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = \frac{1}{\beta_n - \alpha_n} \left[ \frac{1}{2\pi i} \oint_{C(\beta_n)} \frac{f(t)}{t - \alpha_n} \, dt - \frac{1}{2\pi i} \oint_{C(\beta_n)} \frac{f(t)}{t - \beta_n} \, dt \right] \\
&= \frac{1}{2\pi i} \oint_{C(\beta_n)} \frac{f(t)}{(t - \alpha_n)(t - \beta_n)} \, dt \\
&\to \frac{1}{2\pi i} \oint_{C(\beta_n)} \frac{f(t)}{(t - \infty)^2} \, dt \\
&= f'(\infty), \text{ a contradiction. } \quad \Box \end{align*} \]

Riemann Mapping Theorem

Let \( D \subseteq \mathbb{C} \) be a simply connected domain (\( \neq \mathbb{C} \)). Then \( \exists \) 1-1 analytic function \( f \) that maps \( D \) to \( 1 + i < 1 \). Moreover, for any \( z \in D \) and any direction \( \alpha \), one can choose that function so that it maps \( z \) to \( 0 \) and \( \alpha \) direction to that positive real axis. Such restrictions lead to the uniqueness.
In the next two subsections, we construct analytic mappings.

Mapping of simply connected domains.

One way is enough, since we can use the inverse function for the other.
Section 5.2  Möbius Transformations

Defined by

\[ w = f(z) = \frac{az + b}{cz + d} \quad (ad - bc) \]

Translation

\[ w = f(z) = z + c \quad (c \in \mathbb{C}) \]

Rotation

\[ w = f(z) = e^{i\varphi} z \quad (\varphi \in \mathbb{R}) \]

Affine Transformation

\[ w = f(z) = az + b \quad (a, b \in \mathbb{C}) \]

\[ a = |a|e^{i\phi}, \quad a \bar{z} = |a|e^{-i\phi} \bar{z} \]

\[ w = z + 1 + \frac{i}{2} \]

Dilation

\[ z \rightarrow \lambda z, \quad \lambda \in \text{real} \]

Inversion

\[ w = f(z) = \frac{1}{z} \]

(a) Lines through 0 \( \Rightarrow \) Lines through 0

\[ z = \rho e^{i\alpha} \Rightarrow w = \frac{1}{\rho} e^{-i\alpha} \]

(b) Lines not through 0 \( \Rightarrow \) Circles

\[ \frac{1}{z} = 0 \quad \infty \to 0 \]
\[\begin{align*}
A\alpha + B\beta &= C \quad (c \neq 0) \quad (A, B, C \in \mathbb{R}) \\
\alpha &= \frac{1}{2} \quad \beta = \frac{1}{2} = \frac{w}{|w|} = \frac{u - i \nu}{u + i \nu} \quad (w = u + i \nu) \\
\gamma &= \frac{u}{w + i \nu} \quad \delta = -\frac{\nu}{u + i \nu} \\
A\alpha + B\beta &= C \quad \Rightarrow \quad u + i \nu = \frac{A}{c} u + \frac{B}{c} \nu = 0.
\end{align*}\]

(c) Circles passing through 0 \Rightarrow lines (opposite to (b)).

(d) Circles not through 0 \Rightarrow circles.

In general,
\[
\frac{a + b}{c + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{c + d}
\]

\[w_1 = c + d, \quad w_2 = \frac{1}{w_1}, \quad w = \left(b - \frac{ad}{c}\right)w_1 + \frac{a}{c}.
\]

**Theorem.** Let \(f\) be a Möbius transformation. Then,

1. \(f\) is the composition of a finite sequence of translations, non-rigid dilatations, rotations, inversions.
2. \(f\) maps \(\mathcal{U}/\{0\}\) to \(\mathcal{U}/\{0\}\), 1-1, onto.
3. \(f\) maps circles and lines to circles and lines.
4. \(f\) is conformal at every point except its pole.

**Theorem.** If \(g, f \in \text{Möbius} \Rightarrow f \circ g \) is Möbius, too.
The cross-ratio technique.

Given a circle or line in $z$-plane and 3 distinct points $z_1, z_2, z_3$ on this circle or line. (one of them can be $z_0$ if this is a line).

\[ w = f(z) \]

Similarly, given 3 points $w_1, w_2, w_3$ on a line or circle $lw$ in the $w$-plane.

Then, the unique Möbius transformation $w = f(z)$ that maps $z_1$ to $w_1$ and $f(z_j) = w_j$ ($j = 1, 2, 3$) is given by

\[
(w, w_1, w_2, w_3) = (z_1, z_1, z_3, z_3)
\]

\[
\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z_1 - z_1)(z_3 - z_3)}{(z_1 - z_3)(z_3 - z_1)}
\]

\[
\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} \overset{z}{\rightarrow} \frac{(z_1 - z_1)(z_3 - z_3)}{(z_1 - z_3)(z_3 - z_1)}
\]
Example: Find a Möbius transformation:

\[ 0 \rightarrow 1, \quad 1 \rightarrow 2, \quad -1 \rightarrow 4 \]

Solution:

\[
\begin{align*}
(w, v, z, t) &= (0, 1, -1) \\
\left(\frac{w-i}{(w-4)(2-i)}\right) &= \frac{-2(w-i)}{(w-4)(2-i)} \\
\therefore w &= \frac{(16-6i)z + 2i}{(6-2i)z + 2} = \frac{(8-3i)z + 1}{(3-i)z + 1}.
\end{align*}
\]

Example: Find a Möbius transformation that maps \( D_1 : |t| > 1 \) onto \( D_2 : \text{Re} w < 0 \)

\[
\begin{align*}
t_1 &= 1, \quad t_2 = -i, \quad t_3 = -1 \\
w_1 &= 0, \quad w_2 = i, \quad w_3 = \infty
\end{align*}
\]

\[
(W, w_1, w_2, w_3) = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{w-w_1}{w_2-w_1}
\]

\[
(t, t_1, t_2, t_3) = \frac{(t-t_1)(t_2-t_3)}{(t-t_3)(t_2-t_1)} = \frac{(2-1)(1-i)}{(2-2)(2-i)} = \frac{(2-i)(1-i)}{(2+i)(-i-1)}
\]

\[
\frac{W}{i} = \frac{(2-i)(1-i)}{(2+i)(-i-1)} \Rightarrow W = \frac{1-z}{1+z}
\]
Example. Let $x_1 \in \mathbb{R}$, $-1 < \alpha < 1$. Consider $f(z)$ such that $f'(z) = (z - x_1)^\alpha$.

\[ f : (\infty, x_1) \rightarrow I. \quad (x_1, \infty) \rightarrow II. \]

If $z \in (x_1, x_1)$, $\arg f'(z) = \arg (z - x_1)^\alpha = \alpha \arg (z - x_1) = \alpha \pi.$

If $z \in (x_1, \infty)$, $\arg f'(z) = \alpha \arg (z - x_1) = \alpha \cdot 0 = 0$

[Neglect multiples of $2\pi$.]

Recall: $g(z)$ analytic, $g'(z_0) \neq 0$ Then, \( \forall z, \; \theta \in \mathbb{C}, \; \theta(z_0) = \theta \cdot e^{\lambda(z_0)} \)

\[ w'(z_0) = \frac{g'(z_0)z'(z_0)}{g'(z_0) + z'(z_0)} \]

Every curve through $z_0$ is rotated through the same angle $\arg g'(z_0)$.

Back to $f'(z) = (z - x_1)^\alpha$. \([z, x_1]\) is a curve. Every point on this curve has tangent $\parallel x$-axis. The image of \([z, x_1]\) under $f$ is also a curve, all of whose tangents make an angle $\alpha \pi$ with the real axis. So, a line.
Example \[ f(z) = \frac{2}{3} z^{3/2}, \quad f'(z) = \frac{2}{3} z^{1/2}. \]

Example Suppose \[ f'(t) = A (z - x_1)^a \quad (x_1 \in \mathbb{R}, -1 < a < 1), \]
where \( A \in \mathbb{C} \).

Schwarz-Christoffel transformations

\[ f'(t) = A (z - x_1)^{\alpha_1} \ldots (z - x_n)^{\alpha_n} \]

\( A \in \mathbb{C}, \quad -\infty < x_1 < x_2 < \ldots < x_n < \infty, \)

all \( \alpha_j \in (-1, 1), \quad j = 1, \ldots, n. \)

\[ f(z) = A \int_0^z (s - x_1)^{\alpha_1} (s - x_2)^{\alpha_2} \ldots (s - x_n)^{\alpha_n} ds + B. \]

\[ \text{arg} (z - x_j) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \]
\[
\arg f'(t) = \arg A + \alpha_1 \arg (t-x_1) + \cdots + \alpha_n \arg (t-x_n).
\]

<table>
<thead>
<tr>
<th>Interval</th>
<th>Angle of image</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, x_1))</td>
<td>(\arg A + \alpha_1 \pi)</td>
</tr>
<tr>
<td>((x_1, x_2))</td>
<td>(\arg A + \alpha_2 \pi)</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>((x_{n-1}, x_n))</td>
<td>(\arg A + \alpha_n \pi)</td>
</tr>
<tr>
<td>((x_n, \infty))</td>
<td>(\arg A)</td>
</tr>
</tbody>
</table>

**Theorem** Let \(P\) be a positively oriented polygon having consecutive corners at \(w_1, w_2, \ldots, w_n\) with corresponding right turn angles \(\phi_j\) (\(j=1, 2, \ldots, n\)). Then there exists a one-to-one conformal map from the upper half-plane onto the interior of \(P\). Moreover, this map can be constructed as the following Schwarz–Christoffel transformation

\[
f(z) = A \int_0^z \left(\frac{5-x_1}{5-x_2}\right)^{\phi_1/\pi} \cdots \left(\frac{5-x_{n-1}}{5-x_n}\right)^{\phi_n/\pi} \, ds + B
\]

with \(-\infty < x_1 < x_2 < \cdots < x_{n-1} < \infty\), and

\[
f(x_1) = w_1, \quad f(x_2) = w_2, \quad \ldots, \quad f(x_n) = w_{n-1}, \quad f(\infty) = w_n.
\]

**Note:** \(\phi_j \in (-\pi, \pi), \quad j=1, 2, \ldots, n\).

\[\phi_1 + \phi_2 + \cdots + \phi_n = -2\pi.\]
Example Determine a Schwartz-Christoffel transformation which maps the upper half-plane onto the semi-infinite strip \(|\text{Re } w | < 1, \text{Im } w > 0\).

\[ f(z) = A \frac{(z - x_1)^{\frac{1}{2}} (z - x_2)^{-\frac{1}{2}}}{(z - x_3)^{\frac{1}{2}} (z - x_4)^{-\frac{1}{2}}} \]

Let \( x_1 = -1, x_2 = 1 \).

\[ f(z) = A \int_{0}^{z} \frac{ds}{s^{\frac{1}{2}} (s+1)^{\frac{1}{2}} (s-1)^{\frac{1}{2}}} + 13 \]

\[ = \frac{A}{i} \int_{0}^{z} \frac{ds}{\sqrt{1-s^2}} + 13 \]

\[ = \frac{A}{i} \sin^{-1} z + 13 \]

Setting \( f(-1) = w_1 = -1 \): \(-i A \sin^{-1}(-1) + 13 = -1\)

Setting \( f(1) = w_2 = 1 \): \(-i A \sin^{-1}(1) + 13 = 1\)

\[ \Rightarrow 13 = 0, A = 2i / \pi. \]

\[ f(z) = \frac{1}{\pi} \sin^{-1} z. \]