Review for Final Exam

Chapter 1. Complex Analysis

1. Complex numbers, real and imaginary parts, module, argument. The $n$th roots of the unity.
2. Complex functions, limits and continuity: equivalence with the real and imaginary parts.
3. Derivative. The Cauchy–Riemann equations. Conjugate harmonic functions. Concept: analytic in a domain and at a point. Continuous everywhere and nowhere differentiable functions: $f(z) = |z|$, $\text{Re } z$, etc.
4. Elementary functions: $e^z$, $\sin z$, $\cos z$, $\tan z$, $\log z$, and $z^\alpha$. Is $\sin z$ a bounded function?
5. Concept of branch.

Chapter 2. Complex Integration

1. Definition of a smooth arc (or curve) and its parametrization, contours, simple contours, loops (i.e., closed contours), orientations.
2. Definition of contour integration. Some basic properties (e.g., linearity, bounds, etc.).

Chapter 3. Series Representations

4. Order of zero of an analytic function in a domain. Simple zeros. If $z_0$ is a zero of $f$ of order $m \geq 1$, then $f(z) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$. True or false: If $f(z)$ is analytic in $|z| < 1$ and $f(z) = 0$ for all $z$ with $z = x \in \mathbb{R}$ and $0 < x < 1$, then $f = 0$ in $|z| < 1$.
5. Singularities: removable, pole of order $m$, and essential. Equivalence for $f(z)$ to have a pole at $z_0$ of order $m$: $f(z) = g(z)/(z - z_0)^m$ for some analytic $g$ with $g(z_0) \neq 0$.

Chapter 4. Residue Theory

1. Definition of residue. Calculation of residue by Laurent series. Calculation of the residue of a removable singularity and that of a pole of order $m$.
2. Cauchy’s Residue Theorem and application to evaluation of integrals.
3. Techniques of integration using the residue theorem, different types of integrals. (Practice!)

Chapter 5. Conformal Mappings

1. Harmonic functions and its invariance.
2. Definition of a conformal mapping. Conformal if analytical with a nonzero derivative.
4. Möbius transforms, their constructions using the cross product.
Part 2. Ordinary Differential Equations and Dynamical Systems

Chapter 1. Introduction

1. Concept: a system of first-order ordinary differential equations (ODEs), initial conditions and initial-value problems, linear systems, autonomous systems, etc. Reformulation of a high-order single equation into a system of first-order ODEs. Motion of a pendulum.

2. Definition of a solution. Phase space, solution trajectory, and associated flow. Why trajectories do not cross each other? How to determine the direction of a trajectory?


Chapter 2. Linear Systems

1. Plane system $\dot{x} = Ax$. Solution method. Classify the solution $x(t) = 0$: stable node (sink); unstable node (source); saddle (which is unstable); stable spiral; unstable spiral; and center. Stability diagram.

2. General linear system $\dot{x} = Ax$: the unique solution $x(t) = e^{At}x(0)$ for all $t$. Solution structure: each $x_i(t)$ is a linear combinations of $te^{\alpha t}\cos(\beta t)$ and $te^{\alpha t}\sin(\beta t)$.

Chapter 3. Nonlinear Systems

1. Statement of existence and uniqueness of solution. Equivalence: $\dot{x} = f(x,t)$ and $x(0) = x_0$ if and only if $x(t) = x(0) + \int_0^t f(x(s),s)\,ds$. What is a Lipschitz function? Maximal solution interval and finite-time blow up. Example: $\dot{x} = x^2$ and $x(0) = x_0$. Continuous dependence of solution on data and parameters.

2. Critical points and equilibrium solutions. Linearized system and linear stability of a critical points. Find the matrix $A = Df(x^*)$ at a critical point $x^*$ of $f$.

3. Definition of stability, instability, and asymptotic stability of a critical point. Stable, unstable, and center manifolds. Stability of a hyperbolic critical point is determined by its linear stability. (For a non-hyperbolic critical point, see the example on page 41 of the lecture note.) Liapunov function for a critical point: definition, and relation to the stability.

4. Gradient system $\ddot{x} = -\nabla U(x)$. Trajectories are orthogonal to level surfaces of potential $U$. (Why?) Monotonic decay of the potential energy for a non-constant solution: $U(x(t_2)) < U(x(t_1))$ if $t_1 > t_2$. (Why?) No constant periodic solutions. A locally strict minimum of $U$ is an asymptotically stable critical point of $U$. (Why?)

5. Hamiltonian system: $\dot{x} = \partial_y H$ and $\dot{y} = -\partial_x H$. Conservation of energy: $(d/dt)H(x(t),y(t)) = 0$. (Liouville’s Theorem: Any Hamiltonian flow preserves the volume in the phase space.) Example: the mechanical system of $n$ particles $x_1, \ldots, x_n$ interacting through a potential $U = U(x_1, \ldots, x_n)$. Phase portrait of the (rescaled) motion of a frictionless pendulum: $\ddot{\theta} + \sin \theta = 0$.

6. Concept of closed orbits (or cycles) and limit cycles, their stability, instability, and asymptotic stability. Poincaré–Bendixson Theorem. The trapping region method. Poincaré maps.