Math 210B: Mathematical Methods for Physical Sciences and Engineering
Winter quarter 2018

Review for Final Exam

Part 1. Complex Analysis

Chapter 1. Analytic Functions

1. Complex numbers, real and imaginary parts, module, argument. The nth roots of the unity.
2. Complex functions, limits and continuity: equivalence with the real and imaginary parts.
3. Derivative. The Cauchy–Riemann equations. Conjugate harmonic functions. Concept: analytic in a domain and at a point. Continuous everywhere and nowhere differentiable functions: $f(z) = |z|$, $\operatorname{Re} z$, etc.
4. Elementary functions: e^z, $\sin z$, $\cos z$, $\tan z$, $\log z$, and z^α. Is $\sin z$ a bounded function? Complex functions as mappings: pre-images (i.e., domains) and images (i.e., ranges).
5. Concept of branch.

Chapter 2. Complex Integration

1. Definition of a smooth arc (or curve) and its parametrization, contours, simple contours, loops (i.e., closed contours), orientations.
2. Definition of contour integration. Some basic properties (e.g., linearity, bounds, etc.).

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt = F(z_{\mathrm{T}}) - F(z_{\mathrm{I}}) \qquad \text{with } F'(z) = f(z).$$

3. Path independence. Deformation Invariance Theorem. Cauchy's Integration Theorem.
4. Simply connected domains. Cauchy's Integration Formula and its generalization. Analyticity implies the infinite-differentiability.
5. Liouville's Theorem. Mean-Value Theorem. Maximum Principle. Applications.

Chapter 3. Series Representations

1. Power series, radius of convergence. Term-by-term limit, integration, and differentiation.
2. Taylor series: definition and coefficients. Maclaurin series of some elementary functions.
3. Laurent series, coefficients. Term-by-term limit, integration, and differentiation.
4. Order of zero of an analytic function in a domain. Simple zeros. If z_0 is a zero of f of order $m \ge 1$, then $f(z) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$. True or fals: If $f(z)$ is analytic in $|z| < 1$ and $f(z) = 0$ for all z with $z = x \in \mathbb{R}$ and $0 < x < 1$, then $f = 0$ in $|z| < 1$.
5. Singularities: removable, pole of order m, and essential. Equivalence for $f(z)$ to have a pole at z_0 of order m: $f(z) = g(z)/(z - z_0)^m$ for some analytic g with $g(z_0) \neq 0$.

Chapter 4. Residue Theory

1. Definition of residue. Calculation of residue by Laurent series. Calculation of the residue of a removable singularity and that of a pole of order m.
2. Cauchy's Residue Theorem and application to evaluation of integrals.
3. Techniques of integration using the residue theorem, different types of integrals. (Practice!)

Chapter 5. Conformal Mappings

1. Harmonic functions and its invariance.
2. Definition of a conformal mapping. Conformal if analytical with a nonzero derivative.
3. Riemann Mapping Theorem.
4. Möbius transforms, their constructions using the cross product.

Part 2. Ordinary Differential Equations and Dynamical Systems

Chapter 1. Introduction

1. Concept: a system of first-order ordinary differential equations (ODEs), initial conditions and initial-value problems, linear systems, autonomous systems, etc. Reformulation of a high-order single equation into a system of first-order ODEs. Motion of a pendulum.
2. Definition of a solution. Phase space, solution trajectory, and associated flow. Why trajectories do not cross each other? How to determine the direction of a trajectory?
3. Review of single ODEs. Solution method for separable equation, first-order linear equation, and second-order linear and homogeneous equation with constant coefficients. Critical points and their stabilities for a single autonomous equation.
4. Review of linear algebra: matrix eigenvalues and eigenvectors, diagonalization, and Jordan forms. Matrix exponential.

Chapter 2. Linear Systems

1. Plane system $\dot{\mathbf{x}} = A\mathbf{x}$. Solution method. Classify the solution $\mathbf{x}(t) = \mathbf{0}$: stable node (sink); unstable node (source); saddle (which is unstable); stable spiral; unstable spiral; and center. Stability diagram.
2. General linear system $\dot{\mathbf{x}} = A\mathbf{x}$: the unique solution $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$ for all t. Solution structure: each $x_i(t)$ is a linear combinations of $t^n e^{\alpha t}\cos(\beta t)$ and $t^n e^{\alpha t}\sin(\beta t)$.
3. General linear system $\dot{\mathbf{x}} = A\mathbf{x}$: Stability of the critical point $\mathbf{x} = \mathbf{0}$. Rules of thumb: asymptotically stable if real parts of all eigenvalues of A are all negative, and unstable if the real part of one of the eigenvalues of A is positive. Invariant subspaces E^{s}, E^{u}, and E^{c}; $\mathbb{R}^n = E^{\mathrm{s}} \oplus E^{\mathrm{u}} \oplus E^{\mathrm{c}}$.

Chapter 3. Nonlinear Systems

1. Statement of existence and uniqueness of solution. Equivalence: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ and $\mathbf{x}(0) = \mathbf{x}_0$ if and only if $\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{f}(\mathbf{x}(s), s)\, ds$. What is a Lipschitz function? Maximal solution interval and finite-time blow up. Example: $\dot{x} = x^2$ and $x(0) = x_0$. Continuous dependence of solution on data and parameters.
2. Critical points and equilibrium solutions. Linearized system and linear stability of a critical points. Find the matrix $A = D\mathbf{f}(\mathbf{x}^*)$ at a critical point $\mathbf{x}^*$ of $\mathbf{f}$.
3. Definition of stability, instability, and asymptotic stability of a critical point. Stable, unstable, and center manifolds. Stability of a hyperbolic critical point is determined by its linear stability. (For a non-hyperbolic critical point, see the example on page 41 of the lecture note.) Liapunov function for a critical point: definition, and relation to the stability.
4. Gradient system $\dot{\mathbf{x}} = -\nabla U(\mathbf{x})$. Trajectories are orthogonal to level surfaces of potential U. (Why?) Monotonic decay of the potential energy for a non-constant solution: $U(\mathbf{x}(t_2)) < U(\mathbf{x}(t_1)$ if $t_1 > t_1$. (Why?) No constant periodic solutions. A locally strict minimum of U is an asymptotically stable critical point of U. (Why?)
5. Hamiltonian system: $\dot{\mathbf{x}} = \partial_{\mathbf{y}} H$ and $\dot{\mathbf{y}} = -\partial_{\mathbf{x}} H$. Conservation of energy: $(d/dt)H(\mathbf{x}(t), \mathbf{y}(t)) = 0$. (Liouville's Theorem: Any Hamiltonian flow preserves the volume in the phase space.) Example: the mechanical system of n particles $x_1, \ldots, x_n$ interacting through a potential $U = U(x_1, \ldots, x_n)$. Phase portrait of the (rescaled) motion of a frictionless pendulum: $\ddot{\theta} + \sin\theta = 0$.
6. Concept of closed orbits (or cycles) and limit cycles, their stability, instability, and asymptotic stability. Poincaré–Bendixson Theorem. The trapping region method. Poincaré maps.