Math 210B: Mathematical Methods in Physical Sciences and Engineering
Winter quarter, 2018

Homework Assignment 6
Due Monday, February 26, 2018

1. For each of the following ODEs, find the general solution, and then find the specific solution using the initial conditions if they are given:

(1) \( ty' + 12y = t^3 \) \((t > 0)\);
(2) \( \frac{dy}{dt} = \frac{1 + y^2}{ty + t^2y} \);
(3) \( y'' + 6y' + 13y = 0 \) and \( y(0) = 0, y'(0) = -8 \).

2. Consider the differential equation \( \frac{dy}{dt} = 10y (1 - y^2) \) \((0 \leq t < \infty)\).

(1) Find all the equilibrium solutions and classify each of them as asymptotically stable or unstable as \( t \to +\infty \).
(2) Sketch the solutions with the initial values \( y(0) = -2, -0.5, 0.5, 2 \), respectively. Indicate their monotonicity and limiting behavior as \( t \to \infty \).

3. Rewrite the initial-value problem \( x''' - 2(\sin t)x'' + xx' = 1 + t^2 \) and \( x(0) = 1, x'(0) = 2 \) and \( x''(0) = -1 \) for \( x = x(t) \) into the corresponding initial-value problem for a system of 3 first-order equations, first in the form of individual equations and then in the form of a vector equation.

4. Diagonalize the matrix \( A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \) and solve the system of equations \( x' = Ax \).

5. Find all \( \alpha \in \mathbb{R} \) so that \( \lim_{t \to +\infty} x(t) = 0 \) for any solution \( x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \) of \( x' = \begin{bmatrix} -1 & \alpha \\ 1 & -1 \end{bmatrix} x \).

6. Determine the stability of the fixed point \( x^* = 0 \) and draw the phase portrait for the plane system \( x' = Ax \) for each of the following \( A \):

\( \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \); \( \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \); \( \begin{bmatrix} 5 & 2 \\ -17 & -5 \end{bmatrix} \); \( \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \).

7. Find the invariant space \( E^s, E^u, \) and \( E^c \) for the flow \( x(t) = e^{At} \), where \( A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \).

8. For each of the following systems, find the fixed points, classify them, and sketch neighboring trajectories:

(1) \( \dot{x} = x - y \) and \( \dot{y} = x^2 - 4 \);
(2) \( \dot{x} = xy - 1 \) and \( \dot{y} = x - y^3 \).