Math 210B: Mathematical Methods in Physical Sciences and Engineering
Winter quarter, 2018

Homework Assignment 6

Due Monday, February 26, 2018

1. For each of the following ODEs, find the general solution, and then find the specific solution using the initial conditions if they are given:
   (1) $ty' + 12y = t^3 \quad (t > 0)$;
   (2) $\dfrac{dy}{dt} = \dfrac{1+y^2}{ty+t^2y}$;
   (3) $y'' + 6y' + 13y = 0$ and $y(0) = 0$, $y'(0) = -8$.

2. Consider the differential equation $dy/dt = 10y\,(1-y^2)$ $(0 \le t < \infty)$.
   (1) Find all the equilibrium solutions and classify each of them as asymptotically stable or unstable as $t \to +\infty$.
   (2) Sketch the solutions with the initial values $y(0) = -2,\ -0.5,\ 0.5,\ 2$, respectively. Indicate their monotonicity and limiting behavior as $t \to \infty$.

3. Rewrite the initial-value problem $x''' - 2(\sin t)x'' + xx' = 1 + t^2$ and $x(0) = 1$, $x'(0) = 2$ and $x''(0) = -1$ for $x = x(t)$ into the corresponding initial-value problem for a system of 3 first-order equations, first in the form of individule equations and then in the form of a vector equation.

4. Diagonalize the matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ and solve the system of equations $\mathbf{x}' = A\mathbf{x}$.

5. Find all $\alpha \in \mathbb{R}$ so that $\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{0}$ for any solution $\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ of $\mathbf{x}' = \begin{bmatrix} -1 & \alpha \\ 1 & -1 \end{bmatrix}\mathbf{x}$.

6. Determine the stability of the fixed point $\mathbf{x}^* = \mathbf{0}$ and draw the phase portrait for the plane system $\mathbf{x}' = A\mathbf{x}$ for each of the following A :

$$(1)\ \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}; \qquad (2)\ \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}; \qquad (3)\ \begin{bmatrix} 5 & 2 \\ -17 & -5 \end{bmatrix}; \qquad (4)\ \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}.$$

7. Find the invariant space E^{s}, E^{u}, and E^{c} for the flow $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$, where $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

8. For each of the following systems, find the fixed points, classify them, and sketch neighboring trajectories:
   (1) $\dot{x} = x - y$ and $\dot{y} = x^2 - 4$;
   (2) $\dot{x} = xy - 1$ and $\dot{y} = x - y^3$.