Math 210B: Mathematical Methods in Physical Sciences and Engineering
Winter quarter, 2018

Homework Assignment 7

Due Wednesday, March 7, 2018

1. Solve each of the following initial-value problems; find the maximal interval of the solution, and calculate the limit of the solution as t approaches each of the end points of this interval.
   (1) $\dot{x} = -1/(2x)$ and $x(0) = 1$;
   (2) $\dot{x} = x^{-2}$ and $x(1/3) = 1$;
   (3) $\dot{x}_1 = -x_2/x_3^2$, $\dot{x}_2 = x_1/x_3^2$, $\dot{x}_3 = 1$, and $x_1(1/\pi) = 0$, $x_2(1/\pi) = -1$, $x_3(1/\pi) = 1/\pi$.

2. Show that the set $S = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1^2/4\}$ is invariant under the flow ϕ_t defined by the nonlinear system $\dot{x}_1 = -x_1$ and $\dot{x}_2 = x_1^2 + 2x_2$.

3. For each of the following nonlinear systems, find all the critical points; and for each of the critical points, find the linearized system and classify the critical point as a sink (stable node), source (unstable node), or saddle:
   (1) $\dot{x}_1 = x_1 - x_1x_2$ and $\dot{x}_2 = x_2 - x_1^2$;
   (2) $\dot{x}_1 = -4x_2 + 2x_1x_2 - 8$ and $\dot{x}_2 = 4x_2^2 - x_1^2$.

4. Use appropriate Liapunov functions to determine the stability of the equilibrium points of the following systems:
   (1) $\dot{x}_1 = -x_1 + x_2 + x_1x_2$ and $\dot{x}_2 = x_1 - x_2 - x_1^2 - x_2^3$;
   (2) $\dot{x}_1 = -x_1 - 2x_2 + x_1x_2^2$ and $\dot{x}_2 = 3x_1 - 3x_2 + x_2^3$.

5. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Suppose $\mathbf{x}_0$ is a strict local minimum of $V(\mathbf{x})$. Show that $V(\mathbf{x}) - V(\mathbf{x}_0)$ is a strict Liapunov function for the gradient system $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$.

6. Show that the function $V(x_1, x_2) = x_1^2(x_1 - 1)^2 + x_2^2$ has a strict local minima at $(0, 0)$ and $(1, 0)$, and a saddle point at $(1/2, 0)$. Show further that the gradient system $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$ has stable nodes at $(0, 0)$ and $(1, 0)$, and a saddle at $(1/2, 0)$.

7. For each of the following Hamiltonians, sketch on the same phase plane the phase portraits for the Hamiltonian system $\dot{x} = \partial_y H$ and $\dot{y} = -\partial_x H$, and the gradient system $\dot{x} = -\partial_x H$ and $\dot{y} = -\partial_y H$ orthogonal to the Hamiltonian system:
   (1) $H(x, y) = x^2 + 2y^2$;
   (2) $H(x, y) = x^2 - y^2$.