Math 210B: Mathematical Methods in Physical Sciences and Engineering
Winter quarter, 2018

Homework Assignment 8

Due Friday, March 16, 2018

1. Consider $\dot{x} = y^2 + y\cos x$, $\dot{y} = 2xy + \sin x$. Determine if this is a gradient system. If it is, then find the potential $V = V(x, y)$ so that $\dot{x} = -\partial_x V$ and $\dot{y} = -\partial_y V$.

2. Consider the Lorenz system

$$\dot{x} = \sigma(y - x), \quad \dot{y} = \rho x - y - xz, \quad \dot{z} = xy - \beta z,$$

where σ, ρ, and β are all positive constants. Prove the following:

   (1) This system is invariant under the transformation $(x, y, z, t) \to (-x, -y, z, t)$.
   (2) The z-axis is invariant under the flow of this system.
   (3) The system has the critical points at $(0, 0, 0)$ and $(\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1)$ for $\rho > 1$.
   (4) For $\rho \in (0, 1)$, the function $V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma z^2$ is a Liapunov function for the critical point $(0, 0, 0)$.

3. Show by Dulac's Criterion that the system $\dot{x} = x(2 - x - y)$, $\dot{y} = y(4x - x^2 - 3)$ has no closed orbits in the first quadrant $x > 0$, $y > 0$.

4. Consider the system $\dot{x} = x - y - x(x^2 + 2y^2)$, $\dot{y} = x + y - y(x^2 + 2y^2)$.

   (1) Write the system in the polar coordinates. (You can use $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (\dot{y}x - \dot{x}y)/r^2$.)
   (2) Use the trapping region method to show that this system has a closed orbit in the region defined by $r_1 < r < r_2$ for some positive numbers r_1 and r_2 with $0 < r_1 < r_2$.

5. Consider the system $\dot{x} = x(1 - 4x^2 - y^2) - 0.5y(1 + x)$, $\dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x)$.

   (1) Show that the origin is an unstable fixed point.
   (2) Consider $\dot{V}$ with $V(x, y) = (1 - 4x^2 - y^2)^2$ to show that all trajectories approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

6. Consider the vector field given in the polar coordinates by $\dot{r} = r - r^2$, $\dot{\theta} = 1$.

   (1) Let S be the positive x-axis and compute the Poincaré map from S to itself.
   (2) Show that the system has a unique periodic orbit and classify its stability.