Math 210C Homework 1 Solutions

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Problem 1. For each of the following partial differential equations, find out its order and determine if it is a linear equation:
Solution.
(1) $u_x + uu_y = 0$ with $u = u(x, y)$;
First order and nonlinear since uu_y is not linear.

(2) $u_t - \Delta u = u(u - 1/2)(u - 1)$ with $u = u(x, t)$ and $x \in \mathbb{R}^n$;
Second order and nonlinear since $u(u - 1/2)(u - 1)$ is nonlinear.

(3) $\Delta u = \sin |x|$ with $u = u(x)$ and $x \in \mathbb{R}^n$;
Second order and linear since $\sin |x|$ is not a function of u.

(4) $\Delta u - \sinh(u) = 0$ with $u = u(x, y, z)$;
Second order and nonlinear since $\sinh(u)$ is not linear in u.

(5) $u_{tt} - \Delta u + \Delta^2 u = 0$ with $u = u(x, t)$ and $x \in \mathbb{R}^n$;
Fourth order due to Δ^2 and linear.

Problem 2. Let $u \in C^2(\mathbb{R}^n)$. Prove that $\nabla \cdot \nabla u = \Delta u$.

Proof. Standard fact.

□

Problem 3. Let $\Omega = (0, 1) \times (0, 1)$. Show that there exists no $u \in C^2(\bar{\Omega})$ such that $\Delta u = 1$ in Ω and $u = 0$ on $\partial\Omega$.

Proof. The proof is contained in Page 10 of Professor's notes .

□

Problem 4. Verify that $u_n(x, y) = \sin(nx) \sinh(ny)$ is a solution to $\Delta u = 0$ for any $n > 0$.

Proof. We have $[\sin(nx)]'' = -n^2 \sin(nx)$ and $[\sinh(ny)]'' = n^2 \sinh(ny)$. Hence $\Delta u = u_{xx} + u_{yy} = (-n^2 + n^2) \sin(nx) \sinh(ny) = 0$.

□

Problem 5. Let $u = u(x, y)$. Solve $xu_x + u = 0$.

Solution. The equation of characteristic curve is

$$\frac{dy}{dx} = \frac{1}{x}$$

hence $y = \ln|x| + C$. It turns out that $u(x, y) = f(y - \ln|x| - C)$ for some differentiable function f.

Problem 6. Let $u = u(x, y)$ and solve the equation $3u_y + u_{xy} = 0$.

Solution. Let $v = u_y$. We have $3v + v_x = 0$, which implies $(e^{3x} \cdot v)_x = 0$. Hence $e^{3x}v(x, y) = f(y)$ for an arbitrary function f. Now integrating v with respect to y we have

$$u(x, y) = \int e^{-3x} f(y)dy = g(y)e^{-3x} + h(x)$$

where h is an arbitrary function of x which is the integral constant when integrating with respect to y.

Problem 7. Let $u = u(x, y)$. Find the general solution to the equation $u_x + 2xy^2u_y = 0$.

Solution. Consider a new variable t so that the characteristic curve is $x = x(t)$ and $y = y(t)$. Then we have $x'(t) = 1$, $y'(t) = 2xy^2$ and $u_t = x'(t)u_x + y'(t)u_y = 0$. Solving the equation

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = 2xy^2$$

we get $dy/y^2 = 2xdx$. Hence the characteristic curve is $-1/y = x^2 + C$ and u is a constant along the curve. Hence $u(x, y) = f(x^2 + 1/y)$ for some differentiable function f.

Problem 8. Let $u = u(x, y)$. Solve the equation $\sqrt{1 - x^2}u_x + u_y = 0$ with $u(0, y) = e^{-y^2}$.

Solution. Like previous problem we write down $x'(t) = \sqrt{1 - x^2}$ and $y'(t) = 1$. Hence

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

The solution to the above eequation is $y = \arcsin(x) + C$. hence the solution is $u(x, y) = f(y - \arcsin(x))$. Plugging in $u(0, y) = f(y - 0) = e^{-y^2}$, we have

$$u(x, y) = e^{-(y - \arcsin(x))^2}.$$