Math 210C Homework 2 Solutions

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Problem 1. Consider the initial-value problem

\[
\begin{cases}
u_x + yu_y = 0 & \text{for } x, y \in \mathbb{R} \\
u(x, 0) = \phi(x) & \text{for } x \in \mathbb{R}
\end{cases}
\]

Prove the following:

(1) If \( \phi(x) = x \) \((x \in \mathbb{R})\) then this initial-value problem does not have a solution;

(2) If \( \phi(x) = 1 \) \((x \in \mathbb{R})\) then this initial-value problem has many solutions.

Proof. The characteristic curve for this equation satisfies \( \frac{dy}{dx} = y \), hence these are a family of exponential curves

\[ y = Ce^x \quad C \in \mathbb{R} \]

hence the initial condition is given on the characteristic curve \( C = 0 \). For the equation to be solvable \( \phi \) needs to be constant. Hence if \( \phi(x) = x \) then this equation has no solution, but as \( \phi(x) = 1 \) we can construct many solutions like \( u(x, y) = f(ye^{-x}) \) where \( f \) is a differentiable function satisfying \( f(0) = 1 \).

Problem 2. . Solve the following Cauchy problems of first-order equations:

(1) \( u_x + xu_y - u_z = u \) and \( u(x, y, 1) = x + y; \)

Solution. We find that the characteristic curve satisfies \( \dot{x} = 1, \dot{y} = x \) and \( \dot{z} = -1 \). Let \( w(s) = u(x(s), y(s), z(s)) \) and we have \( \dot{w} = w \). Hence the curve starting from the initial hypersurface is

\[ (x(s), y(s), z(s)) = (x_0 + s, y_0 + \frac{1}{2} s^2 + x_0 s, 1 - s) \]

and along the characteristic curve we have \( w = w(0)e^s \). We have \( s = 1 - z \) and as we plug this into \( x(s) \) and \( y(s) \) we have

\[ (x_0, y_0) = (x + z - 1, y - (x + \frac{z - 1}{2})(1 - z)) \]

Hence we have \( w(0) = x_0 + y_0 = xz + y + \frac{z^2 - 1}{2} \). Finally we have

\[ u(x, y, z) = w(0)e^{1-z} = (xz + y + \frac{z^2 - 1}{2})e^{1-z}. \]
(2) $u u_x + y u_y = x$ and $u(x, 1) = 2x$.

Solution. We find that the characteristic curve satisfies $\dot{x} = u$ and $\dot{y} = y$. Let $z(s) = u(x(s), y(s))$ and we have $\dot{z} = \dot{u} = x$. Hence we have

$$\begin{cases} x(s) = \frac{x_0 + z_0}{2} e^s + \frac{x_0 - z_0}{2} e^{-s} \\ y(s) = y_0 e^s \\ z(s) = \frac{x_0 + z_0}{2} e^s - \frac{x_0 - z_0}{2} e^{-s} \end{cases}$$

We set $s = 0$ on initial hypersurface $(x_0, y_0) = (x_0, 1)$, hence $e^s = y$ and by the fact that $z_0 = u(x_0, 1) = 2x_0$ we get from the first equation on $x(s)$:

$$x_0 = \frac{z_0}{2} = \frac{2x}{3y + \frac{1}{y}}$$

Plugging this into the equation of $z(s)$ we get

$$u(x, y) = z(s) = \frac{3x}{3y + \frac{1}{y}} - \frac{x}{3y + \frac{1}{y}} = \frac{3y^2 - 1}{3y^2 + 1} x.$$  

**Problem 3.** Classify each of the following second-order equations into elliptic, hyperbolic, or parabolic equations:

(1) $2u_{xx} - 6u_{xy} + 3u_{yy} = 0$;
Hyperbolic, since $2 \cdot 3 - (\frac{6}{2})^2 < 0$.

(2) $u_{xx} - 4u_{xy} + 4u_{yy} = 0$;
Parabolic, since $1 \cdot 4 - (\frac{4}{2})^2 = 0$.

(3) $2u_{xx} - u_{xy} + 3u_{yy} = 0$;
Elliptic, since $2 \cdot 3 - (\frac{1}{2})^2 > 0$.

**Problem 4.** Solve Laplace’s equation on the upper half plane $\{(x, y) : y > 0\}$ with the boundary condition $u(x, 0) = 1$ ($-\infty < x < \infty$).

Solution. The answer is highly nonunique. For example we can choose a linear combination of harmonic polynomials which contains no single $x^n$ terms (e.g. we take $3x^2y - y^3$ instead of $x^3 - 3xy^2$), so that they take value of zero on $x$-axis, then adding 1 will give a solution to this problem.

**Problem 5.** Let $a$ and $b$ be two positive numbers with $a < b$. Solve Poisson’s equation $\Delta u = 1$ in $a < r < b$ with the boundary condition $u = 0$ on the circles $r = a$ and $r = b$, where $r = \sqrt{x^2 + y^2}$.

Solution. Rewriting the Laplacian in polar coordinates we have

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 1$$
Since the boundary condition does not depend on $\theta$, if we define $\tilde{u}(r,\theta) = u(r,\theta + \theta_0)$ as a rotation of the original solution, then $\tilde{u} = u = 0$ on the boundary, and $\Delta(\tilde{u} - u) = 0$. Hence $\tilde{u} = u$ i.e. $u$ is independent of $\theta$. Hence $u(r,\theta) = f(r)$ and $f$ satisfies

\[
\begin{cases}
\frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) = 1 \\
f(a) = f(b) = 0
\end{cases}
\]

Solving the above ODE we have

\[f(r) = \frac{r^2}{2} + c \log(r) + d\]

where $c$ and $d$ are constants adjusted to the boundary conditions $f(a) = f(b) = 0$. Finally we have $u(r,\theta) = f(r)$.

**Problem 6.** Solve the one-dimensional eigenvalue problem $-u'' = \lambda u$ on $(0, L)$ and $u'(0) = u'(L) = 0$ for some given $L > 0$.

**Solution.** Multiply each side by $u$ and from integration by part we see

\[\int_0^L -u''u = -u'u|_0^L + \int_0^L (u')^2 = \int_0^L (u')^2 = \lambda \int_0^L u^2\]

hence $\lambda \geq 0$ and $\lambda = 0$ means $u$ is a constant. When $\lambda > 0$ we try $u(x) = A \sin(kx) + B \cos(kx)$ where $A, B$ and $k$ are constants. Plugging this into the equation we have $k^2 = \lambda$ and

\[kA \cos(k \cdot 0) - kB \sin(k \cdot 0) = kA \cos(kL) - kB \sin(kL) = 0\]

Hence $kA = 0$ and $kL = n\pi$ for $n \in \mathbb{Z}$. Hence the solution is

\[u(x) = B \cos \left( \frac{n\pi x}{L} \right)\]

with $\lambda = n^2 \pi^2 / L^2$ for $n \in \mathbb{Z}$.

**Problem 7.** Use the method of separation of variables to solve the following boundary-value problem of Laplace’s equation:

\[
\begin{cases}
 u_{xx} + u_{yy} = 0 & \text{for } 0 < x < \pi \text{ and } 0 < y < \pi \\
 u_x(0, y) = 0 \text{ and } u_x(\pi, y) = 0 & \text{for } 0 < y < \pi \\
 u(x, 0) = 0 \text{ and } u(x, \pi) = g(x) & \text{for } 0 < x < \pi
\end{cases}
\]

where $g(x)$ is a given, continuous function on $[0, \pi]$.

**Solution.** Suppose $u(x, y) = X(x)Y(y)$ for some functions $X$ and $Y$. Using Laplace equation we have $X''Y + XY'' = 0$, hence

\[- \frac{X''}{X} = \frac{Y''}{Y} = \lambda\]
for some constant $\lambda$. Together with the boundary condition for $u_x$, it follows from **Problem 6** that

$$X(x) = A_n \cos(nx) \quad n \in \mathbb{Z}.$$ 

This also means $\lambda = n^2 \geq 0$. Hence solving $Y$ we have $Y(y) = \sinh(ny)$ for $n > 0$, and $Y(y) = y$ for $n = 0$. To determine these coefficients we use orthogonality of the set $\{\cos(nx) | n \in \mathbb{Z}\}$. Representing $u$ as

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos(nx) \sinh(ny)$$

We take sum only over $\mathbb{Z}^+$ because of the parity of $\cos$ and $\sinh$. Plugging in $u(x, \pi) = g(x)$ we have $g(x) = A_0 \pi + \sum_n A_n \sinh(n \pi) \cos(nx)$. We have

$$\int_0^\pi g(x) \cos(mx) dx = \int_0^\pi \left[ A_0 \pi + \sum_{n=1}^{\infty} A_n \sinh(n \pi) \cos(nx) \right] \cos(mx) dx$$

$$= \int_0^\pi A_0 \pi \cos(mx) dx + \sum_{n=1}^{\infty} A_n \sinh(n \pi) \int_0^\pi \cos(nx) \cos(mx) dx$$

$$= \begin{cases} \frac{\pi}{2} A_m \sinh(m \pi) & m \geq 1 \\ A_0 \pi^2 & m = 0 \end{cases}$$

Hence

$$u(x, y) = \frac{\int_0^\pi g(x) dx}{\pi^2} y + \sum_{n=1}^{\infty} 2 \frac{\int_0^\pi g(x) \cos(nx) dx}{\pi \sinh(n \pi)} \cos(nx) \sinh(y).$$