Math 210C Homework 2 Solutions

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Problem 1. Consider the initial-value problem

$$\begin{cases} u_x + yu_y = 0 & \text{for } x, y \in \mathbb{R} \\ u(x,0) = \phi(x) & \text{for } x \in \mathbb{R} \end{cases}$$

Prove the following:
(1) If $\phi(x) = x$ $(x \in \mathbb{R})$ then this initial-value problem does not have a solution;
(2) If $\phi(x) = 1$ $(x \in \mathbb{R})$ then this initial-value problem has many solutions.

Proof. The characteristic curve for this equation satisfies $\frac{dy}{dx} = y$, hence these are a family of exponential curves

$$y = Ce^x \qquad C \in \mathbb{R}$$

hence the initial condition is given on the characteristic curve $C = 0$. For the equation to be solvable ϕ needs to be constant. Hence if $\phi(x) = x$ then this equation has no solution, but as $\phi(x) = 1$ we can construct many solutions like $u(x,y) = f(ye^{-x})$ where f is a differentiable function satisfying $f(0) = 1$. □

Problem 2. . Solve the following Cauchy problems of first-order equations:
(1) $u_x + xu_y - u_z = u$ and $u(x,y,1) = x + y$;

Solution. We find that the characteristic curve satisfies $\dot{x} = 1$, $\dot{y} = x$ and $\dot{z} = -1$. Let $w(s) = u(x(s), y(s), z(s))$ and we have $\dot{w} = w$. Hence the curve starting from the initial hypersurface is

$$\big(x(s), y(s), z(s)\big) = (x_0 + s, y_0 + \frac{1}{2}s^2 + x_0 s, 1 - s)$$

and along the characteristic curve we have $w = w(0)e^s$. We have $s = 1 - z$ and as we plug this into $x(s)$ and $y(s)$ we have

$$(x_0, y_0) = \big(x + z - 1, y - (x + \frac{z-1}{2})(1-z)\big)$$

Hence we have $w(0) = x_0 + y_0 = xz + y + \frac{z^2-1}{2}$. Finally we have

$$u(x,y,z) = w(0)e^{1-z} = \big(xz + y + \frac{z^2-1}{2}\big)e^{1-z}.$$

(2) $uu_x + yu_y = x$ and $u(x,1) = 2x$.
Solution. We find that the characteristic curve satisfies $\dot{x} = u$ and $\dot{y} = y$. Let $z(s) = u(x(s), y(s))$ and we have $\dot{z} = \dot{u} = x$. Hence we have

$$\begin{cases} x(s) = \frac{x_0+z_0}{2}e^s + \frac{x_0-z_0}{2}e^{-s} \\ y(s) = y_0 e^s \\ z(s) = \frac{x_0+z_0}{2}e^s - \frac{x_0-z_0}{2}e^{-s} \end{cases}$$

We set $s = 0$ on initial hypersurface $(x_0, y_0) = (x_0, 1)$, hence $e^s = y$ and by the fact that $z_0 = u(x_0, 1) = 2x_0$ we get from the first equation on $x(s)$:

$$x_0 = \frac{z_0}{2} = \frac{2x}{3y + \frac{1}{y}}$$

Pluging this into the equation of $z(s)$ we get

$$u(x,y) = z(s) = \frac{3x}{3y + \frac{1}{y}} y - \frac{x}{3y + \frac{1}{y}} \frac{1}{y} = \frac{3y^2 - 1}{3y^2 + 1} x.$$

Problem 3. Classify each of the following second-order equations into elliptic, hyperbolic, or parabolic equations:

(1) $2u_{xx} - 6u_{xy} + 3u_{yy} = 0$;
Hyperbolic, since $2 \cdot 3 - \left(\frac{6}{2}\right)^2 < 0$.

(2) $u_{xx} - 4u_{xy} + 4u_{yy} = 0$;
Parabolic, since $1 \cdot 4 - \left(\frac{4}{2}\right)^2 = 0$.

(3) $2u_{xx} - u_{xy} + 3u_{yy} = 0$;
Elliptic, since $2 \cdot 3 - \left(\frac{1}{2}\right)^2 > 0$.

Problem 4. Solve Laplace's equation on the upper half plane $\{(x,y) : y > 0\}$ with the boundary condition $u(x,0) = 1$ $(-\infty < x < \infty)$.

Solution. The answer is highly nonunique. For example we can choose a linear combination of harmonic polynomials which contains no single x^n terms (e.g. we take $3x^2y - y^3$ instead of $x^3 - 3xy^2$), so that they take value of zero on x-axis, then adding 1 will give a solution to this problem.

Problem 5. Let a and b be two positive numbers with $a < b$. Solve Poisson's equation $\Delta u = 1$ in $a < r < b$ with the boundary condition $u = 0$ on the circles $r = a$ and $r = b$, where $r = \sqrt{x^2 + y^2}$.

Solution. Rewriting the Laplacian in polar coordinates we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 1$$

Since the boundary condition does not depend on θ, if we define $\tilde{u}(r,\theta) = u(r,\theta+\theta_0)$ as a rotation of the original solution, then $\tilde{u} = u = 0$ on the boundary, and $\Delta(\tilde{u} - u) = 0$. Hence $\tilde{u} = u$ i.e. u is independent of θ. Hence $u(r,\theta) = f(r)$ and f satisfies

$$\begin{cases} \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) = 1 \\ f(a) = f(b) = 0 \end{cases}$$

Solving the above ODE we have

$$f(r) = \frac{r^2}{2} + c\log(r) + d$$

where c and d are constants adjusted to the boundary conditions $f(a) = f(b) = 0$. Finally we have $u(r,\theta) = f(r)$.

Problem 6. Solve the one-dimensional eigenvalue problem $-u'' = \lambda u$ on $(0,L)$ and $u'(0) = u'(L) = 0$ for some given $L > 0$.

Solution. Multiply each side by u and from integration by part we see

$$\int_0^L -u''u = -u'u\Big|_0^L + \int_0^L (u')^2 = \int_0^L (u')^2 = \lambda \int_0^L u^2$$

hence $\lambda \geq 0$ and $\lambda = 0$ means u is a constant. When $\lambda > 0$ we try $u(x) = A\sin(kx) + B\cos(kx)$ where A, B and k are constants. Plugging this into the equation we have $k^2 = \lambda$ and

$$kA\cos(k \cdot 0) - kB\sin(k \cdot 0) = kA\cos(kL) - kB\sin(kL) = 0$$

Hence $kA = 0$ and $kL = n\pi$ for $n \in \mathbb{Z}$. Hence the solution is

$$u(x) = B\cos\left(\frac{n\pi x}{L}\right)$$

with $\lambda = n^2\pi^2/L^2$ for $n \in \mathbb{Z}$.

Problem 7. Use the method of separation of variables to solve the following boundary-value problem of Laplace's equation:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } 0 < x < \pi \text{ and } 0 < y < \pi \\ u_x(0,y) = 0 \text{ and } u_x(\pi,y) = 0 & \text{for } 0 < y < \pi \\ u(x,0) = 0 \text{ and } u(x,\pi) = g(x) & \text{for } 0 < x < \pi \end{cases}$$

where $g(x)$ is a given, continuous function on $[0,\pi]$.

Solution. Suppose $u(x,y) = X(x)Y(y)$ for some functions X and Y. Using Laplace equation we have $X''Y + XY'' = 0$, hence

$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda$$

for some constant λ. Together with the boundary condition for u_x, it follows from **Problem 6** that

$$X(x) = A_n \cos(nx) \qquad n \in \mathbb{Z}.$$

This also means $\lambda = n^2 \geq 0$. Hence solving Y we have $Y(y) = \sinh(ny)$ for $n > 0$, and $Y(y) = y$ for $n = 0$. To determine these coefficients we use orthogonality of the set $\{\cos(nx) | n \in \mathbb{Z}\}$. Representing u as

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos(nx) \sinh(ny)$$

We take sum only over $\mathbb{Z}^+$ because of the parity of cos and sinh. Pluging in $u(x, \pi) = g(x)$ we have $g(x) = A_0\pi + \sum_n A_n \sinh(n\pi) \cos(nx)$. We have

$$\begin{aligned}
\int_0^{\pi} g(x) \cos(mx) dx &= \int_0^{\pi} \left[A_0\pi + \sum_{n=1}^{\infty} A_n \sinh(n\pi) \cos(nx)\right] \cos(mx) dx \\
&= \int_0^{\pi} A_0\pi \cos(mx) dx + \sum_{n=1}^{\infty} A_n \sinh(n\pi) \int_0^{\pi} \cos(nx) \cos(mx) dx \\
&= \begin{cases} \frac{\pi}{2} A_m \sinh(m\pi) & m \geq 1 \\ A_0 \pi^2 & m = 0 \end{cases}
\end{aligned}$$

Hence

$$u(x, y) = \frac{\int_0^{\pi} g(x) dx}{\pi^2} y + \sum_{n=1}^{\infty} \frac{2 \int_0^{\pi} g(x) \cos(nx) dx}{\pi \sinh(n\pi)} \cos(nx) \sinh(y).$$