Math 210C Homework 3 Solutions

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Problem 1. For each integer $n \geq 1$, define $u_n,\ v_n : \mathbb{R}^2 \to \mathbb{R}^2$ in the polar coordinates by $u_n(r,\theta) = r^n \cos(n\theta)$ and $v_n(r,\theta) = r^n \sin(n\theta)$. Verify that both u_n and v_n are harmonic functions in $\mathbb{R}^2$.

Solutions. The Laplacian in polar coordinates is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Hence

$$\begin{aligned}
\Delta u_n &= \frac{\partial^2 r^n}{\partial r^2}\cos(n\theta) + \frac{1}{r}\frac{\partial r^n}{\partial r}\cos(n\theta) + \frac{r^n}{r^2}\frac{\partial^2 \cos(n\theta)}{\partial \theta^2} \\
&= \left[n(n-1)r^{n-2} + nr^{n-2} - n^2 r^{n-2}\right]\cos(n\theta) \\
&= 0
\end{aligned}$$

Similar computation works for v_n.

Problem 2. Let $u \in C^2(\mathbb{R}^2)$. Let $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ be smooth functions that define a bijective map from $\mathbb{R}^2$ to $\mathbb{R}^2$ with a smooth inverse $x = x(\xi,\eta)$ and $y = y(\xi,\eta)$. Define $v(\xi,\eta) = u(x,y)$ with $x = x(\xi,\eta)$ and $y = y(\xi,\eta)$ for for any $(\xi,\eta) \in \mathbb{R}^2$.

(1) Verify that

$$\Delta u(x,y) = v_{\xi\xi}|\nabla_{(x,y)}\xi|^2 + v_{\eta\eta}|\nabla_{(x,y)}\eta|^2 + 2v_{\xi\eta}\nabla_{(x,y)}\xi\cdot\nabla_{(x,y)}\eta + v_\xi\Delta_{(x,y)}\eta + v_\eta\Delta_{(x,y)}\eta$$

Proof. Using chain rule we can show this by direct compututation. □

(2) Use Part (1) to show that the Laplacian in the polar coordinates (r,θ) is given by

$$\Delta_{(x,y)}u(r,\theta) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Proof. We have $r = \sqrt{x^2+y^2}$ and $\cos\theta = x/r$, hence plugging these into the equation in (1) we get the result. □

Problem 3. Let $n \geq 2$ be an integer and $r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$ with $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Let $u \in C^2(\mathbb{R})$.

(1) Express Δu in terms of $u'(r)$ and $u''(r)$.

Proof. Using a similar computation as in **Problem 2.**, we have that

$$\Delta u = u_{rr} + \frac{n-1}{r} u_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}} u.$$

Hence in case $u = u(r)$, we have

$$u = u_{rr} + \frac{n-1}{r} u_r.$$

□

(2) Solve $\Delta u = 0$ by finding the general solution to the ordinary differential equation for $u = u(r)$.
Solution. The ODE is that

$$u_{rr} + \frac{n-1}{r} u_r = 0.$$

To solve this we multiply r^{n-1} to both sides, hence $(r^{n-1} u_r)_r = 0$ which implies

$$u(r) = \begin{cases} C \log(r) + C' & \text{when } n = 2 \\ C r^{2-n} + C' & \text{when } n \geq 3 \end{cases}$$

Problem 4. Solve Laplace's equation $\Delta u = 0$ on the disk $r < 1$ with the boundary condition $u = 1 + 3 \sin\theta - 4 \cos(5\theta)$ at $r = 1$.

Solution. By Problem 1, we simply write down

$$u(r, \theta) = 1 + 3r \sin(\theta) - 4r^5 \cos(5\theta).$$

Since each term is harmonic, u is harmonic. By the uniqueness of solution we see u solves Laplace's equation uniquely in the unit disk.

Problem 5. (1) Show that $K(x) = -|x|/2$ satisfies $-K'' = \delta$ in $\mathbb{R}$, where δ is the one-dimensional Dirac delta function at 0. This means that

$$\int_{-\infty}^{\infty} K(x) \phi''(x) dx = -\phi(0)$$

for any continuously differentiable and compactly supported function $\phi = \phi(x)$.

Proof. Using integration by parts we get

$$\begin{aligned}\int_{-\infty}^{\infty} K(x)\phi''(x)dx &= K\phi'\Big|_{-\infty}^{\infty} - \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right)K'(x)\phi'(x)dx \\ &= 0 - \int_{-\infty}^{0} \frac{1}{2}\phi'(x)dx + \int_{0}^{\infty} \frac{1}{2}\phi'(x)dx \\ &= -\phi(0)\end{aligned}$$

□

(2) Construct the Green's function $G = G(x, y)$ on a finite interval (a, b) as

$$G(x, y) = K(x - y) + g^x(y) \qquad \forall y \in (a, b)$$

where g^x for each $x \in (a, b)$ is a harmonic function such that $G(x, a) = G(x, b) = 0$.

Solution. A harmonic function on an interval is a linear function in y, hence $g^x(y) = c_1(x) + c_2(x)y$, therefore we solve $c_1(x)$ and $c_2(x)$ under boundary condition as follows:

$$\begin{cases} K(x-a) + c_1(x) + c_2(x)a = 0 \\ K(x-b) + c_1(x) + c_2(x)b = 0 \end{cases}$$

Hence

$$\begin{cases} c_1(x) = \frac{K(x-a)b - K(x-b)a}{a-b} \\ c_2(x) = \frac{K(x-a) - K(x-b)}{b-a} \end{cases}$$

Hence the answer follows.

Problem 6. (1) Let $v = v(x, y)$ be a harmonic function. Prove that $u(x, y) = v(x^2 - y^2, 2xy)$ is also a harmonic function.

Proof. Using complex analysis, let $i = \sqrt{-1}$, $z = x + iy$ and $\bar{z} = x - iy$ then $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, hence $\Delta v(z, \bar{z}) = 4\partial_{\bar{z}}\partial_z v(z, \bar{z})$. We also have $z^2 = x^2 - y^2 + 2ixy$, hence $u(z, \bar{z}) = v(z^2, \bar{z})$ and

$$\Delta u(z, \bar{z}) = \Delta v(z^2, \bar{z}) = 4\partial_{\bar{z}}\partial_z v(z^2, \bar{z}) = 2z\Delta v\Big|_{(z^2, \bar{z})} = 0$$

Since $\partial_{\bar{z}} z = 0$ and the chain rule.

□

(2) Prove that the mapping $(x, y) \to (x^2 - y^2, 2xy)$ maps the first quadrant $x > 0$ and $y > 0$ to the upper half plan $y > 0$.

Proof. Also from the discussion above, since $z \to z^2$ is a bijection from the first quadrant to upper half plane, the result follows. □

(3) Use the mapping in Part (2) and the Green's function for the upper half plane to construct the Green's function for the first quadrant.

Solution. Recalling that the Green's function for the upper half plane is

$$G(z_1, z_2) = \Phi(z_2 - z_1) - \Phi(z_2 - \bar{z_1}) = \frac{1}{2\pi}(\ln|z_2 - z_1| - \ln|z_2 - \bar{z_1}|)$$

where $z_j = x_j + iy_j = (x_j, y_j)$, $j = 1, 2$. Hence $g^{z_1}(z_2) = G(z_1, z_2)$ is harmonic in z_2 and vanishes when z_2 is on x-axis. By (1) and (2) we see $g^{z_1}(z_2^2)$ is also harmonic in z_2 and vanishes on the boundary of first quadrant, hence the Green's function is

$$\hat{G}(z_1, z_2) = \Phi(z_2^2 - z_1) - \Phi(z_2^2 - \bar{z_1}) = \frac{1}{2\pi}(\ln|z_2^2 - z_1| - \ln|z_2^2 - \bar{z_1}|).$$

Using coordinates (x, y) we have

$$\hat{G}((x_1, y_1), (x_2, y_2)) = \frac{1}{4\pi} \ln \frac{(x_2^2 - y_2^2 - x_1)^2 + (2x_2y_2 - y_1)^2}{(x_2^2 - y_2^2 - x_1)^2 + (2x_2y_2 + y_1)^2}$$

(4) Solve Laplace's equation $\Delta u = 0$ in the region $x > 0$ and $y > 0$ with the boundary conditions $u(0, y) = g(y)$ for $y > 0$ and $u(x, 0) = h(x)$ for $x > 0$, where g and h are given continuous functions.

Solution. Use Green's Formula we have

$$u(x, y) = \int_0^\infty h(\xi) \frac{\partial \hat{G}}{\partial \eta}\Big|_{((x,y),(\xi,0))} d\xi + \int_0^\infty g(\eta) \frac{\partial \hat{G}}{\partial \xi}\Big|_{((x,y),(0,\eta))} d\eta$$

since the outward normal on the boundary of the first quadrant are $-\partial_\xi$ and $-\partial_\eta$ respectively.