Math 210C Homework 4 Solutions

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Problem 1. Let $\alpha > 0$ and define $K(r) = (1/r)e^{-\alpha r}$ $(r > 0)$. Verify that

$$-\Delta K(|x|) + \alpha^2 K(|x|) = 0 \qquad \forall x \in \mathbb{R}^3, x \neq 0.$$

Optional: Prove that $K(|x|)/(4\pi)$ is the fundamental solution to $-\Delta + \alpha$ in $\mathbb{R}^3$, i.e.,

$$-\Delta K + \alpha^2 K = 4\pi\delta \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad K(\infty) = 0.$$

Proof. Using the Laplacian in spherical form, i.e.

$$\Delta K = \frac{\partial^2 K}{\partial r^2} + \frac{2}{r}\frac{\partial K}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^2} K$$

where $\Delta_{\mathbb{S}^2}$ is the Laplacian operator on unit sphere. Since K is a radial function, the last term drops. It is then direct to check that K satisfies the equation.

To see K is the fundamental solution, take any $u : \mathbb{R}^3 \to \mathbb{R}$ as a compactly supported test function. Then if we take $B_\epsilon(0)$ to be a small ball of radius ϵ around the origin, since K satisfies the equation above where $x \neq 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} (-\Delta K(x) + \alpha^2 K(x))u(x)dx \\
&= \int_{B_\epsilon(0)} (-\Delta K(x) + \alpha^2 K(x))u(x)dx \\
&= \int_{\partial B_\epsilon(0)} -\frac{\partial K}{\partial n} u(x) d_{\mathbb{S}^2}(x) + \int_{B_\epsilon(0)} [\nabla K \cdot \nabla u + \alpha^2 K(x)u(x)]dx \\
&= \int_{\partial B_\epsilon(0)} \frac{1+\alpha\epsilon}{\epsilon^2} e^{-\alpha\epsilon} u(x) d_{\mathbb{S}^2}(x) + \int_{B_\epsilon(0)} [\nabla K \cdot \nabla u + \alpha^2 K(x)u(x)]dx
\end{aligned}$$

Now we take $\epsilon \to 0$, and we see that

$$\begin{aligned}
&\int_{\partial B_\epsilon(0)} \frac{1+\alpha\epsilon}{\epsilon^2} e^{-\alpha\epsilon} u(x) d_{\mathbb{S}^2}(x) - 4\pi u(0) \\
&= \int_{\partial B_\epsilon(0)} \frac{(1+\alpha\epsilon)e^{-\alpha\epsilon}u(x) - u(0)}{\epsilon^2} d_{\mathbb{S}^2}(x) \\
&= \int_{\partial B_1(0)} \left[(1+\alpha\epsilon)e^{-\alpha\epsilon}u(\epsilon x) - u(0)\right] d_{\mathbb{S}^2}(x) \longrightarrow 0
\end{aligned}$$

since $(1+\alpha\epsilon)e^{-\alpha\epsilon}u(\epsilon x) \to u(0)$ as $\epsilon \to 0$. For the second term we have

$$|\nabla K| = O(\frac{1}{|x|^2}) \quad \text{and} \quad |K| = O(\frac{1}{|x|})$$

also since $|\nabla u|$ and u is bounded near 0, hence

$$\left| \int_{B_\epsilon(0)} [\nabla K \cdot \nabla u + \alpha^2 K(x)u(x)]dx \right| \leq \int_{B_\epsilon(0)} \Big(\frac{C_1}{|x|^2} + \frac{C_2}{|x|}\Big)dx \leq C\epsilon \to 0$$

Hence the original integral equals $4\pi u(0)$, and also $K(\infty) = 0$. Hence K is the fundamental solution to $-\Delta + \alpha$.

□

Problem 2. Let $\alpha > 0$. Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be two given functions. Prove the uniqueness of solution to the Robin boundary-value problem of Poisson's equation in a bounded and smooth domain Ω:

$$\begin{cases} -\Delta u = f & \text{in } \Omega. \\ \partial_n u + \alpha u = g & \text{on } \partial\Omega. \end{cases}$$

Proof. Let u_1 and u_2 be two solutions to the same equation as above, let $u = u_1 - u_2$, then by linearity u satisfies the equation with $f = 0$ and $g = 0$. We have

$$0 = \int_\Omega u\Delta u = \int_{\partial\Omega} u\partial_n u - \int_\Omega |\nabla u|^2 = -\alpha \int_{\partial\Omega} u^2 - \int_\Omega |\nabla u|^2$$

But the right hand side is always nonnegative, hence $u = 0$ on $\partial\Omega$ and $\nabla u = 0$ in Ω, therefore $u = 0$ in Ω, u_1 is the same solution as u_2.

□

Problem 3. Let κ be a positive number. Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be two given functions. Prove the uniqueness of solution to the boundary-value problem:

$$\begin{cases} -\Delta u + \kappa^2 u = f & \text{in } \Omega. \\ u = g & \text{on } \partial\Omega. \end{cases}$$

or

$$\begin{cases} -\Delta u + \kappa^2 u = f & \text{in } \Omega. \\ \partial_n u = g & \text{on } \partial\Omega. \end{cases}$$

Proof. Like **Problem 2** we show that the only solution to the above equations with $f = 0$ and $g = 0$ is the trivial solution. We have

$$0 = \int_\Omega u\Delta u = \int_{\partial\Omega} u\partial_n u - \int_\Omega |\nabla u|^2 = -\int_\Omega |\nabla u|^2$$

because either $u = 0$ or $\partial_n u = 0$ on $\partial\Omega$ in these two equations. Hence u is constant 0.

□

Problem 4. Let Ω be a bounded and smooth domain in $\mathbb{R}^d$ for some $d \geq 2$. Calculate the Euler–Lagrange equation for the functional

$$E[u] = \int_\Omega \frac{1}{2}[|\Delta u|^2 - \ln(1 + |\nabla u|^2)]dx$$

Solution. Take a smooth function v which is compactly supported in Ω. Then

$$\begin{aligned} \frac{d}{dt}E[u+tv]\Big|_{t=0} &= \frac{d}{dt}\int_\Omega \frac{1}{2}[|\Delta(u+tv)|^2 - \ln(1+|\nabla(u+tv)|^2)]dx\Big|_{t=0} \\ &= \int_\Omega \frac{1}{2}[\frac{d}{dt}|\Delta(u+tv)|^2 - \frac{d}{dt}\ln(1+|\nabla(u+tv)|^2)]dx\Big|_{t=0} \end{aligned}$$

we have

$$\frac{d}{dt}|\Delta(u+tv)|^2\Big|_{t=0} = 2\Delta v(\Delta u + t\Delta v)\Big|_{t=0} = 2\Delta v \Delta u$$

and

$$\frac{d}{dt}\ln(1+|\nabla(u+tv)|^2)\Big|_{t=0} = \frac{2\nabla v \cdot \nabla(u+tv)}{1+|\nabla(u+tv)|^2}\Big|_{t=0} = \frac{2\nabla v \cdot \nabla u}{1+|\nabla u|^2}$$

Hence

$$\begin{aligned} \frac{d}{dt}E[u+tv]\Big|_{t=0} &= \int_\Omega [\Delta v \Delta u - \frac{\nabla v \cdot \nabla u}{1+|\nabla u|^2}]dx \\ &= \int_\Omega \Big[\Delta^2 u + \nabla \cdot (\frac{\nabla u}{1+|\nabla u|^2})\Big] v dx \end{aligned}$$

Hence the Euler-Lagrange equation is

$$\Delta^2 u + \nabla \cdot (\frac{\nabla u}{1+|\nabla u|^2}) = 0.$$

Problem 5. Is the statement of the Mean Value Theorem for a harmonic function still true if the sphere is replaced by a cube or a disk is replaced by a square?

Solution. NO. Mean value property relies on the radial symmetry of the domain, which is not true for cubes and squares. To find a counterexample for squares, use the degree 4 polynomial $u(x, y) = x^4 - 6x^2y^2 + y^4$ on $D = [0, 1] \times [0, 1]$. Also see http://www.jstor.org/stable/2974823, **Theorem 2** for a characterization of the region which makes every harmonic satisfy the mean value property.