Math 210C Homework 5 Solutions

Yucheng Tu
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Problem 1. Let $f \in C([0, 1])$ be given. Use the method of separation of variables to find the series expansion of the solution $u = u(x, t)$ to the following initial-boundary-value problem:

$$
\begin{cases}
  u_t = u_{xx} - u & \text{for } 0 < x < 1, t > 0, \\
  u(0, t) = u(1, t) = 0 & \text{for } t > 0, \\
  u(x, 0) = f(x) & \text{for } 0 < x < 1.
\end{cases}
$$

Solution. Suppose $u$ has the following expansion:

$$
u(x, t) = \sum_{n=0}^{\infty} X_n(x) T_n(t).$$

By the equation we have

$$X_n(x) T_n'(t) = X_n''(x) T_n(t) - X_n(x) T_n(t)$$

i.e.

$$\frac{T_n'(t)}{T_n(t)} = \frac{X_n''(x)}{X_n(x)} - 1 = \lambda_n$$

where $\lambda_n$ is a constant depending only on $n$. Combining the boundary condition $u(0, t) = u(1, t) = 0$ with $X_n''(x) = (\lambda_n + 1) X_n(x)$ we have that $\lambda_n + 1 = -n^2 \pi^2$, and

$$X_n(x) = A_n \sin(n\pi x).$$

Solving $T_n'(t) = (-n^2 \pi^2 - 1) T_n(t)$ we have $T_n(t) = B_n e^{-(n^2 \pi^2 + 1)t}$, hence in total

$$u(x, t) = \sum_{n=0}^{\infty} C_n \sin(n\pi x) e^{-(\pi^2 n^2 + 1)t}$$

(1)

where $C_n = A_n B_n$. Plugging in $t = 0$ and using $u(x, 0) = f(x)$ we have that, by Fourier transform

$$C_n = 2 \int_{0}^{1} f(x) \sin(n\pi x) dx$$

(2)

since

$$2 \int_{0}^{1} \sin(n\pi x) \sin(m\pi x) dx = \delta_{mn}.$$
Combining (1) and (2) gives the answer.

**Problem 2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain. Let $u \in C^2(\Omega)$ be a nonzero (real-valued) function and let $\lambda \in \mathbb{R}$. Suppose $-\Delta u = \lambda u$ in $\Omega$ and $u = 0$ on $\partial \Omega$. Show that

$$\lambda = \left( \int_{\Omega} |\nabla u|^2 dx \right) \left( \int_{\Omega} u^2 dx \right)^{-1} > 0.$$  

**Proof.** By integration by parts

$$\int_{\Omega} -\lambda u^2 dx = \int_{\Omega} u \Delta u dx = \int_{\partial \Omega} u \frac{\partial u}{\partial n} dx - \int_{\Omega} |\nabla u|^2 dx = - \int_{\Omega} |\nabla u|^2 dx$$

Since $u$ is nonzero, the result follows.

**Problem 3.** Find the Fourier transform for each of the following functions:

1. $f(x) = (1/2) \chi_{[-1,1]}$, where $\chi_A$ denotes the characteristic function of $A$ (i.e., $\chi_A = 1$ if $x \in A$ and $\chi_A = 0$ if $x \notin A$);

**Solution.** We have

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{2} \chi_{[-1,1]}(x) e^{-i\xi x} dx$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-1}^{1} e^{-i\xi x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{i\xi} - e^{-i\xi}}{2i\xi}$$

$$= \frac{\sin \xi}{\xi \sqrt{2\pi}}.$$

2. (Optional) $f(x) = 1/(1 + x^2)$ ($x \in \mathbb{R}$).

**Solution.** We have

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1 + x^2} e^{-i\xi x} dx$$

consider $g(z) = e^{-i\xi z} / (1 + z^2)$. When $\xi < 0$, we consider $\Gamma^+_{R} = [-R, R] \cup \{ z \in \mathbb{C} : |z| = R \text{ and } \text{Im}(z) > 0 \}$, when $\xi > 0$ we consider $\Gamma^-_{R} = [-R, R] \cup \{ z \in \mathbb{C} : |z| = R \text{ and } \text{Im}(z) < 0 \}$. We also have

$$\text{Res}(g(z); z = i) = \frac{e^{\xi}}{2i} \quad \text{Res}(g(z); z = -i) = \frac{e^{-\xi}}{-2i}.$$
For the case $\xi < 0$, we have
\[ |g(z)| = \left| \frac{1}{1 + z^2 e^{-i\xi(x+iy)}} \right| = \left| \frac{1}{1 + z^2} e^{\xi y} \right| \leq \frac{C}{1 + R^2} \quad \text{for } z \in C_R^+ \]
likewise for $\xi > 0$, we have
\[ |g(z)| = \left| \frac{1}{1 + z^2 e^{-i\xi(x+iy)}} \right| = \left| \frac{1}{1 + z^2} e^{\xi y} \right| \leq \frac{C}{1 + R^2} \quad \text{for } z \in C_R^- \]
From the above we deduce that
\[ \int_{C_R^+} g(z) dz \to 0, \quad \int_{C_R^-} g(z) dz \to 0 \quad \text{as } R \to \infty. \]
Hence
\[ \sqrt{2\pi} \hat{f}(\xi) = \begin{cases} 
\lim_{R \to \infty} \int_{C_R^+} g(z) dz = 2\pi i \cdot \text{Res}(g(z); z = i) = \pi e^{\xi} & (\xi < 0) \\
\lim_{R \to \infty} -\int_{C_R^-} g(z) dz = -2\pi i \cdot \text{Res}(g(z); z = -i) = \pi e^{-\xi} & (\xi > 0) 
\end{cases} \]
Combined with the case $\xi = 0$, where $\sqrt{2\pi} \hat{f}(0) = \pi$ by direct integration, we have
\[ \hat{f}(\xi) = \sqrt{\frac{\pi}{2}} e^{-|\xi|}. \]

**Problem 4.** (1) The Gaussian kernel (or heat kernel) in one space dimension is defined to be
\[ K(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \quad (x \in \mathbb{R}, t > 0) \]
where $D > 0$ is a constant. Verify that $K_t = DK_{xx}$ for all $x \in \mathbb{R}$ and $t > 0$.

Solution. Direct computation.

(2) Define now $u(x, t) = K_n(x, t) = \prod_{i=1}^n K(x_i, t)$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t > 0$. Verify that $u_t = D\Delta u$ for all $x \in \mathbb{R}^n$ and $t > 0$.

Solution. This is because we can separate variables $x_1, \ldots, x_n$ when computing Laplacian.

**Problem 5.** Use the fundamental solution to the heat equation to find a formula of solution to the following initial-boundary-value problem of the heat equation on half-line:
\[ \begin{cases} 
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} & \text{for } x > 0, t > 0, \\
u(0, t) = 0 & \text{for } t > 0, \\
u(x, 0) = \phi(x) & \text{for } x > 0, 
\end{cases} \]
where $D > 0$ is a constant and $\phi$ is a compactly supported continuous function on $[0, \infty)$.

**Solution.** We can extend $\phi$ to $x \in \mathbb{R}$ by setting $\phi(x) = 0$ if $x \leq 0$. Then $u$ is given by

$$u(x, t) = (K * \phi)(x) - (K * \phi)(-x) = \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} \phi(y) \left[ e^{-\frac{(x-y)^2}{4Dt}} - e^{-\frac{(x+y)^2}{4Dt}} \right] dy$$

It is easy to see $u$ satisfies $u_t = D\Delta u$ since $K$ does. Also because the integral of $K$ over $\mathbb{R}$ is 1, and when $t \to 0$, $K(x, t) \to 0$ for $x \neq 0$, we see $K \to \delta$-function. Hence $u(x, 0) = (\delta * \phi)(x) - (\delta * \phi)(-x) = \phi(x) - \phi(-x) = \phi(x)$ for $x > 0$. Finally the construction shows $u(0, t) = 0$.

**Problem 6.** Let $\Omega$ be a bounded and smooth domain in $\mathbb{R}^d$ and denote by $n$ the unit exterior normal to the boundary $\partial \Omega$. Suppose $u = u(x, t)$ ($x \in \Omega, t \geq 0$) is a smooth and bounded function. Suppose also that

$$\begin{cases}
  u_t = D\Delta u & \text{for } x \in \Omega, t > 0, \\
  \frac{\partial u}{\partial n} = 0 & \text{for } x \in \partial \Omega, t > 0, \\
  u(x, 0) = \phi(x) & \text{for } x \in \Omega,
\end{cases}$$

for some $\phi \in C(\Omega)$. Show that

$$\int_{\Omega} u(x, t) dx = \int_{\Omega} \phi(x) dx \quad \forall t > 0.$$

**Proof.** By the smoothness and boundedness of $u$ and $\Omega$, we can differentiate the left hand side as

$$\frac{d}{dt} \int_{\Omega} u(x, t) dx = \int_{\Omega} \frac{\partial u(x, t)}{\partial t} dx = D \int_{\Omega} \Delta u dx = D \int_{\partial \Omega} \frac{\partial u}{\partial n} dx = 0$$

which implies

$$\int_{\Omega} u(x, t) dx = \int_{\Omega} u(x, 0) dx = \int_{\Omega} \phi(x) dx \quad \forall t > 0.$$