Math 210C Homework 5 Solutions

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Problem 1. Let $f \in C([0,1])$ be given. Use the method of separation of variables to find the series expansion of the solution $u = u(x,t)$ to the following initial-boundary-value problem:

$$\begin{cases} u_t = u_{xx} - u & \text{for } 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = f(x) & \text{for } 0 < x < 1. \end{cases}$$

Solution. Suppose u has the following expansion:

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t).$$

By the equation we have

$$X_n(x)T_n'(t) = X_n''(x)T_n(t) - X_n(x)T_n(t)$$

i.e.

$$\frac{T_n'(t)}{T_n(t)} = \frac{X_n''(x)}{X_n(x)} - 1 = \lambda_n$$

where λ_n is a constant depending only on n. Combining the boundary condition $u(0,t) = u(1,t) = 0$ with $X_n''(x) = (\lambda_n + 1)X_n(x)$ we have that $\lambda_n + 1 = -n^2\pi^2$, and

$$X_n(x) = A_n \sin(n\pi x).$$

Solving $T_n'(t) = (-n^2\pi^2 - 1)T_n(t)$ we have $T_n(t) = B_n e^{-(n^2\pi^2+1)t}$, hence in total

$$u(x,t) = \sum_{n=0}^{\infty} C_n \sin(n\pi x) e^{-(\pi^2 n^2 + 1)t} \tag{1}$$

where $C_n = A_n B_n$. Pluging in $t = 0$ and using $u(x,0) = f(x)$ we have that, by Fourier transform

$$C_n = 2\int_0^1 f(x)\sin(\pi n x)dx \tag{2}$$

since

$$2\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \delta_{mn}.$$

Combining (1) and (2) gives the answer.

Problem 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain. Let $u \in C^2(\Omega)$ be a nonzero (real-valued) function and let $\lambda \in \mathbb{R}$. Suppose $-\Delta u = \lambda u$ in Ω and $u = 0$ on $\partial\Omega$. Show that

$$\lambda = \Big(\int_\Omega |\nabla u|^2 dx\Big)\Big(\int_\Omega u^2 dx\Big)^{-1} > 0.$$

Proof. By integration by parts

$$\int_\Omega -\lambda u^2 dx = \int_\Omega u\Delta u dx = \int_{\partial\Omega} u\frac{\partial u}{\partial n} dx - \int_\Omega |\nabla u|^2 dx = -\int_\Omega |\nabla u|^2 dx$$

Since u is nonzero, the result follows.

□

Problem 3. Find the Fourier transform for each of the following functions:

(1) $f(x) = (1/2)\chi_{[-1,1]}$, where χ_A denotes the characteristic function of A (i.e., $\chi_A = 1$ if $x \in A$ and $\chi_A = 0$ if $x \notin A$);

Solution. We have

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{2}\chi_{[-1,1]}(x) e^{-i\xi x} dx \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-1}^{1} e^{-i\xi x} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{e^{i\xi} - e^{-i\xi}}{2i\xi} \\
&= \frac{\sin\xi}{\xi\sqrt{2\pi}}.
\end{aligned}$$

(2) (Optional) $f(x) = 1/(1+x^2)$ $(x \in \mathbb{R})$.

Solution. We have

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1+x^2} e^{-i\xi x} dx$$

consider $g(z) = e^{-i\xi z}/(1+z^2)$. When $\xi < 0$, we consider $\Gamma_R^+ = [-R, R] \cup \{z \in \mathbb{C} : |z| = R \text{ and } \operatorname{Im}(z) > 0\}$, when $\xi > 0$ we consider $\Gamma_R^- = [-R, R] \cup \{z \in \mathbb{C} : |z| = R \text{ and } \operatorname{Im}(z) < 0\}$. We also have

$$\operatorname{Res}(g(z); z = i) = \frac{e^{\xi}}{2i} \qquad \operatorname{Res}(g(z); z = -i) = \frac{e^{-\xi}}{-2i}$$

For the case $\xi < 0$, we have

$$|g(z)| = \left|\frac{1}{1+z^2} e^{-i\xi(x+iy)}\right| = \left|\frac{1}{1+z^2} e^{\xi y}\right| \le \frac{C}{1+R^2} \qquad \text{for } z \in C_R^+$$

likewise for $\xi > 0$, we have

$$|g(z)| = \left|\frac{1}{1+z^2} e^{-i\xi(x+iy)}\right| = \left|\frac{1}{1+z^2} e^{\xi y}\right| \le \frac{C}{1+R^2} \qquad \text{for } z \in C_R^-$$

From the above we deduce that

$$\int_{C_R^+} g(z)dz \longrightarrow 0, \quad \int_{C_R^-} g(z)dz \longrightarrow 0 \text{ as } R \to \infty.$$

Hence

$$\sqrt{2\pi}\hat{f}(\xi) = \begin{cases} \lim\limits_{R\to\infty} \int_{\Gamma_R^+} g(z)dz = 2\pi i \cdot \mathrm{Res}(g(z); z = i) = \pi e^{\xi} & (\xi < 0) \\ \lim\limits_{R\to\infty} -\int_{\Gamma_R^-} g(z)dz = -2\pi i \cdot \mathrm{Res}(g(z); z = -i) = \pi e^{-\xi} & (\xi > 0) \end{cases}$$

Combined with the case $\xi = 0$, where $\sqrt{2\pi}\hat{f}(0) = \pi$ by direct integration, we have

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{2}} e^{-|\xi|}.$$

Problem 4. (1) The Gaussian kernel (or heat kernel) in one space dimension is defined to be

$$K(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \quad (x \in \mathbb{R}, t > 0)$$

where $D > 0$ is a constant. Verify that $K_t = DK_{xx}$ for all $x \in \mathbb{R}$ and $t > 0$.

Solution. Direct computation.

(2) Define now $u(x,t) = K_n(x,t) = \prod_{i=1}^n K(x_i, t)$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$. Verify that $u_t = D\Delta u$ for all $x \in \mathbb{R}^n$ and $t > 0$.

Solution. This is because we can separate variables $x_1, \cdots, x_n$ when computing Laplacian.

Problem 5. Use the fundamental solution to the heat equation to find a formula of solution to the following initial-boundary-value problem of the heat equation on half-line:

$$\begin{cases} u_t = Du_{xx} & \text{for } x > 0, t > 0, \\ u(0,t) = 0 & \text{for } t > 0, \\ u(x,0) = \phi(x) & \text{for } x > 0, \end{cases}$$

where $D > 0$ is a constant and ϕ is a compactly supported continuous function on $[0, \infty)$.

Solution. We can extend ϕ to $x \in \mathbb{R}$ by setting $\phi(x) = 0$ if $x \leq 0$. Then u is given by

$$u(x,t) = (K * \phi)(x) - (K * \phi)(-x) = \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} \phi(y)\big[e^{-\frac{(x-y)^2}{4Dt}} - e^{-\frac{(x+y)^2}{4Dt}}\big]dy$$

It is easy to see u satisfies $u_t = D\Delta u$ since K does. Also because the integral of K over $\mathbb{R}$ is 1, and when $t \to 0$, $K(x,t) \to 0$ for $x \neq 0$, we see $K \to \delta$-function. Hence $u(x,0) = (\delta * \phi)(x) - (\delta * \phi)(-x) = \phi(x) - \phi(-x) = \phi(x)$ for $x > 0$. Finally the construction shows $u(0,t) = 0$.

Problem 6. Let Ω be a bounded and smooth domain in $\mathbb{R}^d$ and denote by n the unit exterior normal to the boundary $\partial\Omega$. Suppose $u = u(x,t)$ $(x \in \Omega, t \geq 0)$ is a smooth and bounded function. Suppose also that

$$\begin{cases} u_t = D\Delta u & \text{for } x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial n} = 0 & \text{for } x \in \partial\Omega, t > 0, \\ u(x,0) = \phi(x) & \text{for } x \in \Omega, \end{cases}$$

for some $\phi \in C(\Omega)$. Show that

$$\int_\Omega u(x,t)dx = \int_\Omega \phi(x)dx \quad \forall t > 0.$$

Proof. By the smoothness and boundedness of u and Ω, we can differentiate the left hand side as

$$\frac{d}{dt}\int_\Omega u(x,t)dx = \int_\Omega \frac{\partial u(x,t)}{\partial t}dx = D\int_\Omega \Delta u dx = D\int_{\partial\Omega} \frac{\partial u}{\partial n}dx = 0$$

which implies

$$\int_\Omega u(x,t)dx = \int_\Omega u(x,0)dx = \int_\Omega \phi(x)dx \quad \forall t > 0.$$

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