Math 210C Homework 6 Solutions

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Problem 1. The Fourier transform for any $u \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx \qquad \forall \xi \in \mathbb{R}^n$$

Prove the following:
(1) For any $h \in \mathbb{R}^n$, $\lambda > 0$, and $u \in L^1(\mathbb{R}^n)$, $\widehat{\tau_h u}(\xi) = e^{-ih\xi}\hat{u}(\xi)$ and $\widehat{\delta_\lambda u}(\xi) = \lambda^n \hat{u}(\lambda\xi)$ $(\xi \in \mathbb{R}^n)$, where $\tau_h u(x) = u(x-h)$ and $\delta_\lambda u(x) = u(x/\lambda)$.

Proof.

$$\begin{aligned} \widehat{\tau_h u}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \tau_h u(x) e^{-ix\cdot\xi} dx, & \widehat{\delta_\lambda u}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \delta_\lambda u(x) e^{-ix\cdot\xi} dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x-h) e^{-ix\cdot\xi} dx & &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x/\lambda) e^{-ix\cdot\xi} dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-i(x+h)\cdot\xi} dx & &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-i\lambda x\cdot\xi} d(\lambda^n x) \\ &= e^{-ih\xi}\hat{u}(\xi) & &= \lambda^n \hat{u}(\lambda\xi) \end{aligned}$$

□

(2) For any $u \in C_c^2(\mathbb{R}^n)$, $\widehat{\Delta u}(\xi) = -|\xi|^2 \hat{u}(\xi)$ $(\xi \in \mathbb{R})$.

Proof.

$$\begin{aligned} \widehat{\Delta u}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \Delta u(x) e^{-ix\cdot\xi} dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} -\nabla u(x) \cdot \nabla_x e^{-ix\cdot\xi} dx & &(\text{supp}(u) \text{ is compact}) \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) \Delta_x e^{-ix\cdot\xi} dx & &(\text{supp}(u) \text{ is compact}) \\ &= -\frac{|\xi|^2}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx \\ &= -|\xi|^2 \hat{u}(\xi) \end{aligned}$$

□

(3) If $u, v \in C_c(\mathbb{R}^n)$, then $\widehat{u * v} = (2\pi)^{\frac{n}{2}} \hat{u}\hat{v}$.

Proof.

$$\begin{aligned}\widehat{u*v}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} u(x-y)v(y)dy \right] e^{-ix\cdot\xi} dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} u(x-y)e^{-i(x-y)\cdot\xi} dx \right] v(y) e^{-iy\cdot\xi} dy \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x)e^{-i(x)\cdot\xi} dx \int_{\mathbb{R}^n} v(y)e^{-iy\cdot\xi} dy \\ &= (2\pi)^{\frac{n}{2}} \hat{u}(\xi)\hat{v}(\xi).\end{aligned}$$

□

Problem 2. Let $D > 0$ and $\alpha > 0$ be two given constants, and consider the diffusion equation

$$u_t = Du_{xx} + \alpha u \qquad (x \in \mathbb{R}, t > 0).$$

Let $k > 0$ and define $u_k(x,t) = e^{\omega t} \sin(kx)(x \in \mathbb{R}, t > 0)$, where ω is a constant to be determined.
(1) Find the formula for $\omega = \omega(k, D, \alpha)$ so that $u_k(x,t)$ solves the above diffusion equation.
(2) With that $\omega = \omega(k, D, \alpha)$, find all $k > 0$ such that $u_k(x,t)$ are bounded as $t \to \infty$.

Solution. Directly plug u_k into the equation we get

$$\omega = \alpha - Dk^2$$

To make u_k bounded, $\omega \leq 0$. Hence $k \geq \sqrt{\alpha/D}$.

Problem 3. Let $u = u(x,t)$ solve the heat equation $u_t = \Delta u$ ($x \in \mathbb{R}^n, t > 0$) with the initial condition $u(x,0) = f(x)$ ($x \in \mathbb{R}^n$).
(1) Let $f \in C(\mathbb{R}^n)$ be bounded. Show that $|u(x,t)| \leq \sup_{y\in\mathbb{R}^n} |f(y)|$ for all $x \in \mathbb{R}^n$ and $t \geq 0$.

Proof. Actually the maximum holds when u satisfy some growth conditions:

$$|u(x,t)| \leq Ae^{B|x|^2} \text{ for some } A, B > 0 \text{ and } \forall t > 0$$

and there exists lots of nonphysical solutions. For the proof of maximum principle see *Partial Differential Equations* by L. Evans.

□

(2) (Optional) Assume in addition $f \in L^1(\mathbb{R}^n)$. Show that $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in \mathbb{R}^n$.

Proof. We have $\max_{x\in\mathbb{R}^n}|K(x,t)| \to 0$ as $t\to\infty$. Hence it follows $|u(x,t)| = |K * f(x,t)| \le \|K\|_\infty \|f\|_1 \to 0$. □

Problem 4. Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, $D > 0$ and $T > 0$ constants, $f \in C(\Omega\times[0,T])$, $g \in C(\partial\Omega\times[0,T])$, and $\phi \in C(\Omega)$. Use the energy method to prove the uniqueness of solution to the initial-boundary-value problem: $u_t - D\Delta u = f$ in $\Omega\times(0,T]$, $u = g$ on $\partial\Omega\times(0,T]$, and $u=\phi$ on $\Omega\times\{0\}$.

Proof. Consider the energy at $t>0$:

$$E(t) = \int_\Omega u^2(x,t)dx$$

Suppose u_1 and u_2 are solutions to the described problem, let $u = u_1 - u_2$, then u has zero boundary conditions. We have

$$\begin{aligned} E'(t) &= \frac{d}{dt}\int_\Omega u(x,t)^2 dx = \int_\Omega 2uu_t dx \\ &= \int_\Omega 2Du\Delta u dx \\ &= -\int_\Omega 2D|\nabla u|^2 dx \le 0 \end{aligned}$$

Hence $E(t)$ is decreasing in t. However $E(0)=0$ and $E(t)\ge 0$ by definition, therefore $E(t) = 0\ \forall t>0$. Hence $u\equiv 0$ and $u_1 = u_2$. □

Problem 5. (The Markov property of solutions to diffusion equations.) Let $u = u(x,t)$ solve the diffusion equation $u_t = D\Delta u$ in $\Omega\times(0,\infty)$, with the zero Dirichlet boundary condition $u(x,t) = 0$ $(x\in\partial\Omega, t>0)$, where $D>0$ is the diffusion constant and Ω is a bounded and smooth domain in $\mathbb{R}^n$. Let $t_1 > 0$ and let $u_1 = u_1(x,t)$ solve the diffusion equation $u_{1t} = D\Delta u_1$ in $\Omega\times(0,\infty)$, with the zero Dirichlet boundary condition $u_1(x,t) = 0$ $(x\in\partial\Omega, t>0)$ and the initial condition $u_1(x,0) = u(x,t_1)$ $(x\in\Omega)$. Prove that $u(x,t_1+t_2) = u_1(x,t_2)$ for any $x\in\Omega$ and any $t_2>0$.

Proof. Set $v(x,t) = u(x,t+t_1) - u_1(x,t)$, then v satisfies the same diffusion equation with zero initial and boundary condition. By the uniqueness of the solution from **Problem 4.** we have $v\equiv 0$. □

Problem 6. Let $D>0$, $\kappa>0$, $Y(x,t) = e^{-\kappa t}K(x,t)$, and $K(x,t) = (4\pi Dt)^{-n/2}e^{\frac{-|x|^2}{4Dt}}$ $(x\in\mathbb{R}^n, t>0)$.

(1) Verify that $Y_t - D\Delta Y + \kappa Y = 0$ in $\mathbb{R}^n\times(0,\infty)$.

Proof. By direct calculation.

□

(2) Let $f \in C(\mathbb{R}^n)$ be bounded. Use the kernel $Y(x,t)$ to find a formula of the solution to the initial-value problem

$$\begin{cases} u_t - \Delta u + \kappa u = 0 \text{ in } \mathbb{R}^n \times (0, \infty), \\ u = f \text{ on } \mathbb{R}^n \times 0. \end{cases}$$

Solution. Taking $D = 1$, we have

$$u(x,t) = f * Y = \int_{\mathbb{R}^n} f(y) Y(x - y, t) dy$$

It is easy to see that u satisfies the equation.