Math 210B. Winter 2014. Lecture Notes Bo Li.

Introduction to Partial Differential Equations (PDE)

Chapter 0 Introduction

[Concepts and examples, boundary conditions and initial conditions. Well posedness. Hadamard example. classical and weak solutions]

Chapter 1. Laplace's Equation / Poisson's Equation

[Solution methods: separation of variables Green's functions. mean-value property. maximum principle. energy method. Fourier transforms. General 2nd order elliptic equation, etc.]

Chapter 2 Heat Equation

[Solution methods: separation of variables Green's function. transform methods, maximum principle. energy method, etc.]

Chapter 3 Wave Equation

[Method of characteristics. transforms, nonhomogeneous problems, energy method, etc.] Classification of 2nd order PDE.

Chapter 4 Techniques for nonlinear PDE

Variational techniques, Hamilton-Jacobi equation, conservation laws, etc.

Chapter 0. Introduction

A PDE is an equation that has partial derivatives of an unknown function of two or more variables (in certain region).

Notation.

$u = u(x_1, x_2, x_3, \cdots, x_n, t)$. $\quad u_t = \frac{\partial u}{\partial t}$, $\quad u_{x_1} = \frac{\partial u}{\partial x_1}$

$$u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j} = \partial^2_{x_i x_j} u = \partial_{x_i} \partial_{x_j} u$$

$u = u(x_1, \cdots, x_n)$; $\quad \nabla u = \begin{bmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix}$ gradient

$x = (x_1, \cdots, x_n)$

$\vec{F}(x) = \begin{bmatrix} F_1(x) \\ \vdots \\ F_n(x) \end{bmatrix}$ Then, the divergence is

$$\nabla \cdot \vec{F}(x) = \sum_{i=1}^{n} \frac{\partial F_i(x)}{\partial x_i}.$$

Laplacian $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \quad (u = u(x_1, \cdots, x_n))$.

Exercise Verify

$$\nabla \cdot \nabla u = \Delta u.$$

$$|\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2}$$

one, two, or three variables

$u = u(x)$, $\quad u = u(x, y)$, or $u = u(x, y, z)$

or $u = u(x_1, x_2, x_3)$

Order of a PDE: highest order of partial derivative of unknown function in the equation.

Linear PDE: linear in the unknown function and all its partial derivatives.

Three most important PDE — linear and order

1. Poisson's (in particular Laplace's) equation

$-\Delta u = f$ Poisson's ($f = f(x)$ is a given function)

$\Delta u = 0$ Laplace's $u = u(x) = u(x_1, \cdots, x_n)$

2. Heat equation or diffusion equation

$u_t - D\Delta u = 0$ (or $= f(x)$)

$D = \text{const.} > 0$ a given (known) constant.

$u = u(x, t) = u(x_1, \cdots, x_n, t)$

$u_t - D\Delta u = f(x)$ inhomogeneous heat equation
f: source term

3. Wave equation

$u_{tt} = c^2 \Delta u$ (or $\underbrace{u_{tt} = c^2\Delta u + f}_{\uparrow}$)

$u = u(x, t) = u(x_1, \cdots, x_n, t)$

inhomogeneous wave equation

More examples of PDE

(1) A simple, linear 1st order equation

$$u_t + a u_x = 0 \qquad u = u(x,t), \quad x \in \mathbb{R}^1$$
$$a = \text{const.}$$

(2) Schrödinger's equation in quantum mechanics

$$i u_t = \Delta u \qquad u = u(x,t) = u(x_1, \cdots, x_n, t)$$
$$i = \sqrt{-1}.$$

(3) Helmholtz's equation

$$\Delta u + k^2 u = 0 \qquad k: \text{ a given constant.}$$

(4) Fokker-Planck equation for diffusion in a potential

$$u_t = \nabla \cdot (D \nabla u + u \nabla \psi)$$

$\psi = \psi(x)$: a given function — the potential.

All these are linear equations. Now, let us look at some nonlinear equations

(5) Eikonal equation

$$|\nabla u(x)| = 1.$$

(6) Burgers' equation $\quad u_t + u u_x = 0$

(7) Nonlinear Poisson's equation

$$-\Delta u = f(u)$$

f is a nonlinear function of u.

e.g., Poisson-Boltzmann equation

(8) (Nonlinear) Reaction-diffusion equation

$$u_t - \Delta u = f(u).$$

(9) Hamilton-Jacobi equation

$$u_t + H(\nabla u, x) = 0$$

(10) Conservation law: $u_t + \nabla \cdot \vec{F}(u) = 0$

Some important systems of PDE is continuum physics

(11) Elasticity equations (linear elasticity)

$$\mu \Delta \vec{u} + (\lambda + \mu) \nabla(\nabla \cdot \vec{u}) = \vec{0}$$

(12) Maxwell's equations

$$\begin{cases} \vec{E}_t = \nabla \times \vec{B} \\ \vec{B}_t = -\nabla \times \vec{E} \\ \nabla \cdot \vec{B} = \nabla \cdot \vec{E} = 0 \end{cases}$$

(13) Navier-Stokes equations for incompressible, viscous flow

$$\rho\left(\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u}\right) = \mu \Delta \vec{u} - \nabla p + \vec{F}$$

$$\nabla \cdot \vec{u} = 0$$

Example Let $u = u(x, y)$. Solve $u_{xy} = 0$.

Integrate in y: $u_x = f(x)$ — $f(x)$ is constant w.r.t. y. $f(x)$ is an arbitrary function of x.

Integrate in x

$$u(x, y) = \int f(x)\,dx + G(y)$$

$G(y)$ is a const. w.r.t. x.

Let $F(x) = \int f(x)\,dx$: an arbitrary function of x. Then

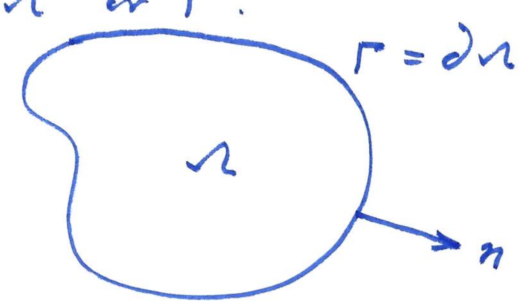
$$u(x, y) = F(x) + G(y).$$

Compare to solutions to a second order, linear, constant coefficient ODE (ordinary differential equation): two arbitrary constants!

To uniquely determine the solution to a PDE, we need some side conditions, in addition to the equations. These are boundary conditions (BC) and initial conditions. (IC)

Example Let $u = u(x, y, z)$, $(x, y, z) \in \Omega$,
Ω: a bounded region in $\mathbb{R}^3$
with boundary $\partial\Omega$ or Γ.

Let $f(x, y, z)$ be a given function defined on $\bar{\Omega}$.
($\bar{\Omega}$: the closure of Ω.
$\bar{\Omega} = \Omega \cup \Gamma$.)



Consider Poisson's equation

$$-\Delta u = f \quad \text{in } \Omega.$$

Dirichlet boundary condition: $u = u_0$ on $\partial\Omega$ where $u_0 = u_0(x, y, z)$ is a given function on the boundary $\partial\Omega$.

Neumann boundary condition: $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$ where $g = g(x, y, z)$ is a given function on $\partial\Omega$.

$\frac{\partial u}{\partial n} = \nabla u \cdot n$ $\quad$ n: unit exterior normal to $\Gamma = \partial\Omega$

Robin (or mixed) boundary condition

$$\frac{\partial u}{\partial n} = a u \quad \text{on } \partial\Omega$$

where $a = a(x, y, z)$ is a given function on $\partial\Omega$.

If we consider the (inhomogeneous) heat equation

$$u_t - D\Delta u = f \quad \text{in } \Omega \times (0, T]$$

where $D > 0$ is a const, $u = u(x, y, z, t)$, $x, y, z \in \Omega$ and $t \in (0, T]$, then we need to imposer one of those three boundary conditions, together with an initial condition:

Initial condition: $u(\underline{x}, 0) = u^{(0)}(x)$, $x \in \partial\Omega$

where $u^{(0)}(x)$ is a given function, x means (x, y, z).

<u>Periodic boundary condition</u> Let $\Omega = (0, L_1) \times (0, L_2)$.
$u : \mathbb{R}^2 \to \mathbb{R}$ satisfies the Ω-periodic boundary condition means

$$\begin{aligned} u(x+L_1, y) &= u(x, y) \\ u(x, y+L_2) &= u(x, y) \end{aligned} \quad \text{for any } (x, y).$$

<u>Cauchy problems</u>. — Examples.

Let $u = u(x, y, t)$.

$$\begin{cases} u_t = \Delta u, & (x, y, t) \in \mathbb{R}^2 \times (0, \infty). \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \mathbb{R}^2 \end{cases}$$

Or:
$$\begin{cases} u_{tt} = \Delta u, & (x, y, t) \in \mathbb{R}^2 \times (0, \infty). \\ u(x, y, 0) = u_0(x, y), & (x, y) \in \mathbb{R}^2 \\ u_t(x, y, 0) = u_1(x, y), & (x, y) \in \mathbb{R}^2 \end{cases}$$

u_0, u_1: given (known).

Goals/objectives of studying PDE (in this class):

1. Solution methods: how to solve PDEs ?
2. Well posedness of PDE: solution existence, solution uniqueness, and continuous dependence on data (initial data, boundary data, source terms, etc.).
3. Qualitative properties of solutions to PDE.
4. More techniques (for nonlinear problems). Concept of weak solutions, properties, etc.

<u>Example</u> (Hadamard) Let $u = u(x, y)$. Consider the Cauchy problem of Laplace's equation

$$(*) \quad \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \\ u = 0, \quad \frac{\partial u}{\partial y} = \frac{1}{n} \sin(nx) & \text{on } \{y = 0\} \end{cases}$$

($n \geq 1$ is an integer)

Verify: $u = \frac{1}{n^2} \sin(nx) \sinh(ny)$ is a solution.

Here $\sinh(ny) = \frac{1}{2}(e^{ny} - e^{-ny})$.

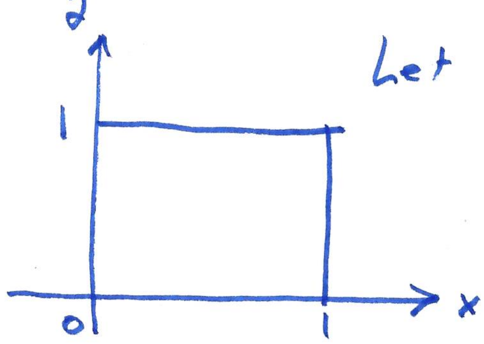
Note
$$(**) \quad \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \\ u = 0, \quad \frac{\partial u}{\partial y} = 0 & \text{on } \{y = 0\} \end{cases}$$
has the solution $u = 0$. in $\mathbb{R}^2$.

Now, the Cauchy data in (*) satisfy, as $n \to \infty$,
$$u_n = 0 \longrightarrow 0 \quad \text{and} \quad \frac{\partial u_n}{\partial y} = \frac{1}{n} \sin(nx) \longrightarrow 0$$

But the solution $u_n = u_n(x, y) = \frac{1}{n^2} \sin(nx) \sinh(ny)$ does NOT converge to the corresponding soln 0:
$$\underbrace{\lim_{n \to \infty} u_n(x, y)}_{\text{may not exist}} \neq 0$$

<u>Conclusion</u> This Cauchy problem of Laplace's equation is ill-posed.

Example



Let Ω = unit square $= (0,1)\times(0,1)$

Prove that there exists no function $u \in C^2(\bar{\Omega})$ such that

$$\begin{cases} \Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (*)$$

Proof. If $u \in C^2(\bar{\Omega})$ (i.e., u, all 1st and 2nd order derivatives of u are continuous on $\bar{\Omega} = [0,1]\times[0,1]$) satisfies $(*)$, then

$$\frac{\partial u}{\partial x} = 0 \quad \text{on} \quad y=0, \ 0 \le x < 1.$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{on} \quad y=0, \ 0 < x < 1.$$

By continuity, $\quad \frac{\partial^2 u}{\partial x^2}(0,0) = 0.$

Similarly, $\quad \frac{\partial^2 u}{\partial y^2}(0,0) = 0.$

Thus, $\quad \Delta u(0,0) = 0.$

But, $\Delta u = 1$ in Ω. By continuity, $\Delta u(0,0) = 1.$

A contradiction!

Weak solution u: $\quad \iint_\Omega \nabla u \cdot \nabla \varphi \, dx\, dy = -\iint_\Omega \varphi \, dx\, dy$

for any $\varphi \in C_0^1(\Omega)$.

Only require ∇u (not Δu) to exist!