Chapter 1 Laplace's Equation and Poisson's Equation

$$\Delta u = 0 \quad \text{and} \quad -\Delta u = f$$

Section 1.1. Method of Separation of Variables

Section 1.2 Green's functions /

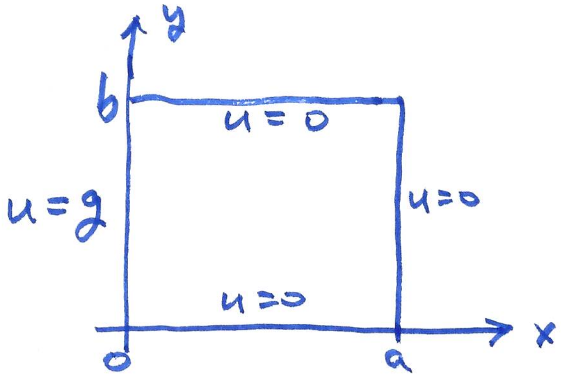
~~Section 1.3~~ Fundamental Solution

Section 1.3 Mean-value Theorems / Maximum Principles

Section 1.4 Energy Method

Section 1.5 General 2nd order elliptic equations

Section 1.1 Method of Separation of Variables



~~Poisson~~. Laplace's eq. on a rectangle $(0,a)\times(0,b)$.

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & 0<x<a,\ 0<y<b \\ u(0,y) = g(y), & 0<y<b \\ u(a,y) = 0 & 0<y<b \\ u(x,0) = 0,\ u(x,b) = 0, & 0<x<a \end{cases}$$

Solution. Let $u(x,y) = h(x)\phi(y)$. $\qquad h(a)=0,\ \phi(0)=\phi(b)=0$.

$$h''(x)\phi(y) + h(x)\phi''(y) = 0$$

$$\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} \equiv \lambda(x,y)$$

$$\lambda = \frac{h''(x)}{h(x)} \Rightarrow \partial_y \lambda = 0, \quad \lambda = -\frac{\phi''(y)}{\phi(y)} \Longrightarrow \partial_x \lambda = 0$$

$$\Longrightarrow \lambda = \text{const.}$$

$$\begin{cases} \phi''(y) + \lambda\phi(y) = 0, & 0<y<b \\ \phi(0) = 0 \\ \phi(b) = 0 \end{cases} \qquad \begin{cases} h''(x) - \lambda h(x) = 0 \\ h(L) = 0. \end{cases}$$

Eigenvalue Problems!

Solve for $\phi(y)$

(1) $\lambda > 0$. $\phi(y) = c_1 \cos(\sqrt{\lambda}\, y) + c_2 \sin(\sqrt{\lambda}\, y)$

$\phi(0) = 0 \Rightarrow c_1 = 0$. $\phi(b) = 0 \Rightarrow \sin(\sqrt{\lambda}\, b) = 0$

$$\Rightarrow \boxed{\begin{array}{l} \lambda = \left(\frac{n\pi}{b}\right)^2 \quad (n = 1, 2, \cdots) \\ \phi(y) = c_n \sin\left(\frac{n\pi y}{b}\right) \quad (n = 1, 2, \cdots) \end{array}}$$

(2) $\lambda = 0$, $\phi''(y) = 0$. $\phi(y) = c_1 + c_2 y$.

$\phi(0) = 0$, $\phi(b) = 0 \Rightarrow c_1 = c_2 = 0$. $\boxed{\phi \equiv 0}$

(3) $\lambda < 0$. $\mu = -\lambda > 0$. $\phi'' - \mu\phi = 0$

$\phi(y) = c_1 e^{\sqrt{\mu} y} + c_2 e^{-\sqrt{\mu} y}$

$\phi(0) = \phi(b) = 0 \Rightarrow c_1 = c_2 = 0$. $\boxed{\phi \equiv 0}$

Solve for $h(x)$

$\lambda = \left(\frac{n\pi}{b}\right)^2$. $\begin{cases} h''(x) - \left(\frac{n\pi}{b}\right)^2 h = 0 \\ h(a) = 0 \end{cases}$

$h(x) = a_1 \cosh \frac{n\pi}{b}(x-a) + a_2 \sinh \frac{n\pi}{b}(x-a)$

$h(a) = 0 \Rightarrow a_1 = 0$. $(\cosh 0 = 1)$

So, $h(x) = a_2 \sinh \frac{n\pi}{b}(x-a)$

$u(x,y) = h(x)\,\phi(y) = c_n \sinh\left(\frac{n\pi}{b}(x-a)\right) \sin\left(\frac{n\pi y}{b}\right)$

$n = 1, 2, \cdots$

Each satisfies the equation $\Delta u = 0$ but not the B.C.

Superposition $u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}(x-a)\right) \sin\left(\frac{n\pi y}{b}\right)$

We need to use the boundary condition $u(0,y) = g(y)$ $(0 < y < b)$ to determine all c_n.

$$g(y) = u(0,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{b}(-a)\right) \sin \frac{n\pi y}{b}$$

Fact: $\left\{ \sin\frac{n\pi y}{b} \right\}_{n=1}^{\infty}$ is an orthogonal system in $L^2(0,b)$.

$$\int_0^b \sin\frac{n\pi y}{b}\sin\frac{m\pi y}{b}\,dy = \begin{cases} 0 & \text{if } m \neq n \\ \frac{b}{2} & \text{if } m = n \end{cases}$$

$$\int_0^b g(y)\sin\frac{m\pi y}{b}\,dy = \sum_{n=1}^{\infty} C_n \sinh\left(\left(-\frac{a}{b}\right)n\pi\right)\int_0^b \sin\frac{n\pi y}{b}\sin\frac{m\pi y}{b}\,dy$$

$$= -C_m \sinh\left(\frac{am\pi}{b}\right)\frac{b}{2}$$

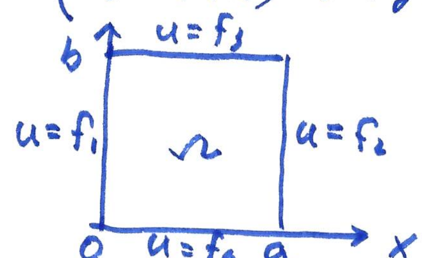
$$C_m = -\frac{2}{b\sinh\left(\frac{am\pi}{b}\right)}\int_0^b g(y)\sin\frac{m\pi y}{b}\,dy \qquad m = 1, 2, \cdots$$

Use $A_m = C_m \sinh\left(-\frac{am\pi}{b}\right)$, $m = 1, 2, \cdots$

$$\boxed{\begin{aligned} A_m &= \frac{2}{b}\int_0^b g(y)\sin\frac{m\pi y}{b}\,dy, \quad m = 1, 2, \cdots \\ u(x,y) &= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{b}. \end{aligned}}$$

Laplace's equation on a rectangle: more general Dirichltet BC.

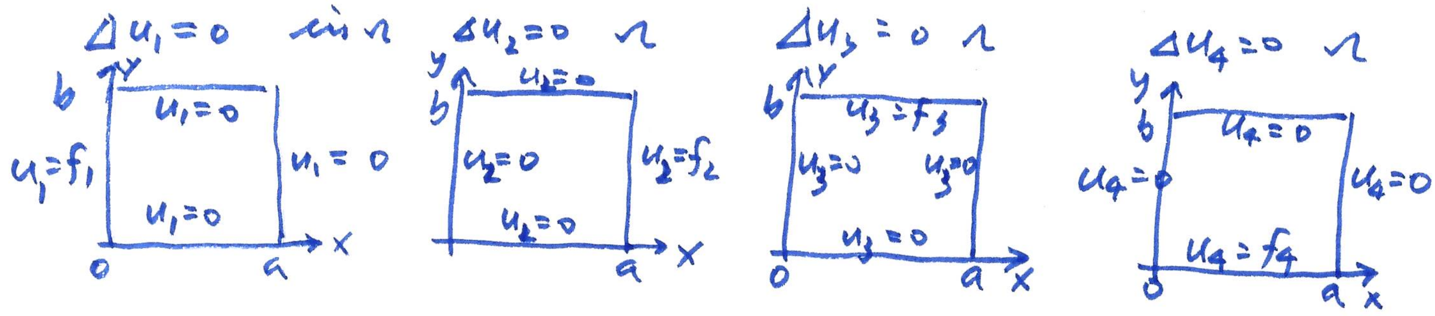
$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 & (0 < x < a,\ 0 < y < b) \\ \text{Dirichlet BC:} & \end{cases}$$



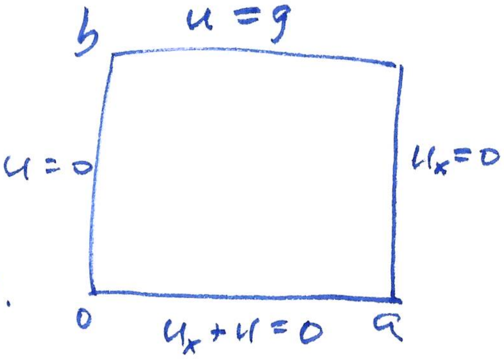
e.g. $u = f_1$ on $x = 0$ means $u(0,y) = f_1(y)$ $(0 < y < b)$.

All f_1, f_2, f_3, f_4 are given (known) functions.

The sol'n is the superposition of sol'ns to four problems: $u = u_1 + u_2 + u_3 + u_4$.



<u>Other boundary conditions</u>



$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad (0<x<a,\ 0<y<b) \\ BC: \quad u(0,y)=0,\ u_x(a,y)=0,\ 0<y<b. \\ \qquad u_x(x,0)+u(x,0)=0,\ u(x,b)=g(x) \\ \qquad\qquad 0<x<a. \end{cases}$$

<u>Solution</u> Let $u(x,y) = X(x)\,Y(y)$. $\dfrac{X''}{X} + \dfrac{Y''}{Y} = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda = \text{const}.$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \quad (0<x<a) \\ X(0) = 0 \quad X'(a) = 0 \end{cases}$$

$$\begin{cases} Y''(y) - \lambda Y(y) = 0 \quad (0<y<b) \\ Y'(0) + Y(0) = 0 \end{cases}$$

Solve for λ an $X = X(x)$.

(1) $\lambda > 0$. $X(x) = c_1 \cos(\sqrt{\lambda}\,x) + c_2 \sin(\sqrt{\lambda}\,x)$

$X(0) = 0 \Rightarrow c_1 = 0$, $X(x) = c_2 \sin(\sqrt{\lambda}\,x)$. $X'(x) = \sqrt{\lambda}\,c_2 \cos(\sqrt{\lambda}\,x)$

$X'(a) = 0 \Longrightarrow \cos(\sqrt{\lambda}\,a) = 0$ $\boxed{\lambda_n = \left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{a^2} \quad n = 0, 1, 2, 3, \cdots}$

$$\boxed{X_n(x) = c_n \sin \frac{\left(n+\frac{1}{2}\right)\pi x}{a} \qquad (n = 0, 1, 2, \cdots)}$$

(2) $\lambda = 0 \Rightarrow X(x) \equiv 0$

(3) $\lambda < 0$. $X(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$

$X(0)=0,\ X'(a)=0 \Longrightarrow X(x) \equiv 0$

Solve for $Y = Y(y)$ with $\lambda = \lambda_n = (n+\frac{1}{2})^2 \pi^2/a^2 = \beta_n^2$

$Y(y) = A\cosh(\beta_n y) + B\sinh(\beta_n y)$. $\beta_n = (n+\frac{1}{2})\pi/a$.

$0 = Y'(0) + Y(0) = B\beta_n + A$

$B = -1. \Longrightarrow A = \beta_n$.

$Y(y) = \beta_n \cosh(\beta_n y) - \sinh(\beta_n y)$. $(n = 0, 1, 2, \cdots)$

$$u(x,y) = \sum_{n=0}^{\infty} A_n \sin\beta_n x\,(\beta_n\cosh\beta_n y - \sinh\beta_n y).$$

To determine all A_n $(n \geq 1)$, we use the b.c.
$u(x,b) = g(x)$, $(0 < x < a)$.

$$g(x) = u(x,b) = \sum_{n=0}^{\infty} A_n \sin\beta_n x\,(\beta_n\cosh(\beta_n b) - \sinh(\beta_n b))$$

Fact $\{\sin\beta_n x\}_{n=0}^{\infty}$ is an orthogonal system in $L^2(0,a)$:

$$\int_0^a \sin\beta_n x\,\sin\beta_m x\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ \frac{a}{2} & \text{if } n = m. \end{cases}$$

So, $$\int_0^a g(x)\sin nx\,dx\,dx = A_n[\beta_n\cosh(\beta_n b) - \sinh(\beta_n b)]\frac{a}{2}.$$

$$\boxed{\begin{aligned} A_n &= \frac{2}{a}(\beta_n\cosh\beta_n b - \sinh\beta_n b)^{-1}\int_0^a g(x)\sin\beta_n x\,dx \\ \beta_n &= (n+\tfrac{1}{2})\pi/a. \quad (n = 0, 1, 2, \cdots) \\ u(x,y) &= \sum_{n=0}^{\infty} A_n \sin\beta_n x\,(\beta_n\cosh\beta_n y - \sin\beta_n y) \end{aligned}}$$

Laplace's equation in a three-dimensional box.

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0. & 0 < x, y, z < \pi. \\ u(\pi, y, z) = g(y, z) & 0 < y, z < \pi \\ u(0, y, z) = u(x, 0, z) = u(x, \pi, z) = u(x, y, 0) = u(x, y, \pi) = 0 \end{cases}$$

$u(x, y, z) = X(x)\, Y(y)\, Z(z).$ $\quad \dfrac{X''}{X} + \dfrac{Y''}{Y} + \dfrac{Z''}{Z} = 0$

$X(0) = Y(0) = Z(0) = Y(\pi) = Z(\pi) = 0$

$\cancel{\dfrac{X''}{X}}, \cancel{\dfrac{Y''}{Y}}, \dfrac{Z''}{Z} = \text{consts.}$

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = \lambda \quad (= \mu$$

$$\frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda, \quad \frac{Y''}{Y} = -\lambda - \frac{Z''}{Z}$$

$$\boxed{\frac{Y''}{Y} = -\mu. \quad \frac{Z''}{Z} = -\lambda + \mu = \sigma. \quad X''/X = \lambda = \mu - \sigma}$$

$Y(y) = \sin(my)$ $(m = 1, 2, \cdots)$

$Z(z) = \sin(nz)$ $(n = 1, 2, \cdots)$

$X'' = (m^2 + n^2) X, \quad X(0) = 0 \Longrightarrow X(x) = A \sinh(\sqrt{m^2+n^2}\, x)$

$$\boxed{u(x, y, z) = \sum_{m,n=1}^{\infty} A_{mn} \sinh(\sqrt{m^2+n^2}\, x) \sin my \sin nz}$$

$$g(y, z) = u(\pi, y, z) = \sum_{m,n=1}^{\infty} A_{mn} \sinh(\sqrt{m^2+n^2}\, \pi) \sin(my) \sin(nz)$$

$\{\sin(my)\sin(nz)\}_{m,n=1}^{\infty}$: orthogonal in $L^2((0, \pi) \times (0, \pi))$.

$$\int_0^\pi \int_0^\pi (\sin(my)\sin(ny))^2\, dy\, dz = \frac{\pi^2}{4}.$$

So,

$$\boxed{A_{mn} = \frac{4}{\pi^2 \sinh(\sqrt{m^2+n^2}\,\pi)} \int_0^\pi \int_0^\pi g(y, z) \sin(my) \sin(nz)\, dy\, dz. \quad (m, n = 1, 2, \cdots)}$$

The Dirichlet boundary-value problem of Laplace's equation on a disk — Poisson's formula

$$(*)\quad \begin{cases} u_{xx} + u_{yy} = 0 & \text{if } x^2+y^2<a^2 \\ u(x,y) = h(\theta) & \text{if } x^2+y^2=a^2 \end{cases} \quad \begin{pmatrix} x = a\cos\theta \\ y = a\sin\theta \end{pmatrix}$$

Here: $a>0$, a const. $h=h(\theta)$, continuous, 2π-periodic.

Polar coordinates $\begin{matrix} x = r\cos\theta \\ y = r\sin\theta. \end{matrix}$ $\quad [\, r_x \neq x_r^{-1} \,]$

$r = r(x,y)$, $\theta = \theta(x,y)$

$$\left.\begin{aligned} 1 &= r_x\cos\theta - r\sin\theta\,\theta_x \\ 0 &= r_y\cos\theta - r\sin\theta\,\theta_y \\ 0 &= r_x\sin\theta + r\cos\theta\,\theta_x \\ 1 &= r_y\sin\theta + r\cos\theta\,\theta_y \end{aligned}\right\} \Rightarrow \left.\begin{aligned} r_x &= \cos\theta \\ r_y &= \sin\theta \\ \theta_x &= -\tfrac{1}{r}\sin\theta \\ \theta_y &= \tfrac{1}{r}\cos\theta \end{aligned}\right\}$$

$$u_x = u_r r_x + u_\theta \theta_x = \cos\theta\, u_r - \frac{1}{r}\sin\theta\, u_\theta$$

$$u_{xx} = -\sin\theta\,\theta_x u_r + \cos\theta\,(u_r)_x - \frac{\cos\theta\,\theta_x r - \sin\theta\, r_x}{r^2} u_\theta$$
$$- \frac{1}{r}\sin\theta\,(u_\theta)_x$$
$$= \frac{1}{r}\sin^2\theta\, u_r + \cos\theta\left(u_{rr}\cos\theta - u_{r\theta}\frac{1}{r}\sin\theta\right)$$
$$+ \frac{1}{r^2}\left(\sin\theta\cos\theta \mp r\cos\theta\left(-\frac{1}{r}\sin\theta\right)\right)u_\theta$$
$$- \frac{1}{r}\sin\theta\left(\cos\theta\, u_{\theta r} + u_{\theta\theta}\left(-\frac{1}{r}\right)\sin\theta\right)$$
$$= \frac{1}{r^2}\sin^2\theta\, u_{\theta\theta} + \cos^2\theta\, u_{rr} - \frac{2}{r}\sin\theta\cos\theta\, u_{r\theta}$$
$$+ \frac{1}{r}\sin^2\theta\, u_r + \frac{2}{r^2}\sin\theta\cos\theta\, u_\theta$$

Similarly

$$u_{yy} = \frac{\cos^2\theta}{r^2} u_{\theta\theta} + \sin^2\theta\, u_{rr} + \frac{2}{r}\sin\theta\cos\theta\, u_{r\theta}$$
$$+ \frac{1}{r}\cos^2\theta\, u_r - \frac{2}{r^2}\sin\theta\cos\theta\, u_\theta$$

$$\boxed{u_{xx}+u_{yy} = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta} \quad \left(=\frac{1}{r}(ru_r)_r+\frac{1}{r^2}u_{\theta\theta}\right)}$$

Now, back to (*) of page $\boxed{17}$.

$u = R(r)\Theta(\theta)$.

$$0 = u_{xx}+u_{yy} = u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}$$
$$= R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}=0$$

$$\frac{r^2R''}{R}+\frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda = \text{const}.$$

$$\begin{cases} \Theta''+\lambda\Theta=0 \\ \Theta: \ 2\pi\text{-periodic} \end{cases}$$

(1) $\lambda > 0$. $\Theta(\theta) = c_1\cos(\sqrt{\lambda}\,\theta)+c_2\sin(\sqrt{\lambda}\,\theta)$

$\Theta(-\pi)=\Theta(\pi) \Longrightarrow \quad c_2\sin(\sqrt{\lambda}\,\pi)=0$

$\sqrt{\lambda}\,\pi = n\pi$. $\boxed{\lambda = 0, 1^2, 2^2, \cdots, n^2, \cdots}$

$\Theta_0(\theta)=\frac{1}{2}A_0$, $\Theta_n(\theta) = (A_n\cos n\theta+B_n\sin n\theta)$

$n=1,2,\cdots$.

$$\begin{cases} r^2R''+rR'-\lambda R=0. \\ R(0) \text{ is finite } [R(r) \text{ is cont. at } r=0]. \end{cases}$$

$\lambda = n^2$ $(n\ge 1)$: $r^2R''+rR'-n^2R=0$

$R(r) = r^\alpha$ $\quad r^2\alpha(\alpha-1)r^{\alpha-2}+r\alpha r^{\alpha-1}-n^2r^\alpha=0$

$\alpha(\alpha-1)+\alpha-n^2=0$. $\alpha=\pm n$

$\lim_{r\to 0} r^{-n}=\infty$. So, $\alpha=n$. $R_n(r)=r^n$.

$\lambda=0$: $r^2R''+rR'=0$ $\quad (rR')'=0$,

$rR'=C_1$ $\quad R'=\frac{C_1}{r}$. $\cancel{R=}$ $R(r)=C_2+C_1\ln r$

$r\to 0$: $R(r)$ is bounded. So, $C_1=0$. $R(r)=C_2$.

$$\boxed{u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)}$$

$$u(a,\theta) = h(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

$\{\sin n\theta, \cos n\theta, 1\}_{n=1}^{\infty}$: orthogonal in $L^2(0,2\pi)$.

$$\boxed{\begin{aligned} A_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos n\phi \, d\phi \quad (n=0,1,2,\cdots) \\ B_n &= \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin n\phi \, d\phi \quad (n=1,2,\cdots) \end{aligned}}$$

Now, derive Poisson's formula.

$$u(r,\theta) = \int_0^{2\pi} h(\phi)\frac{d\phi}{2\pi} + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)\left[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\right] d\phi$$

$$= \int_0^{2\pi} h(\phi)\left\{\underbrace{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi))}\right\}\frac{d\phi}{2\pi}$$

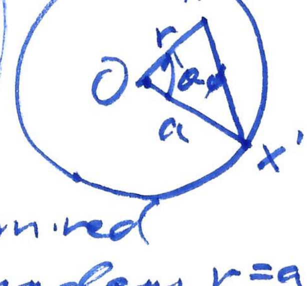
$$= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in(\theta-\phi)}$$

$$= 1 + \frac{re^{i(\theta-\phi)}}{a - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{a - re^{-i(\theta-\phi)}}$$

$$= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}$$

<u>Poisson's formular</u> $$\boxed{u(r,\theta) = \frac{(a^2-r^2)}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\,d\phi}$$

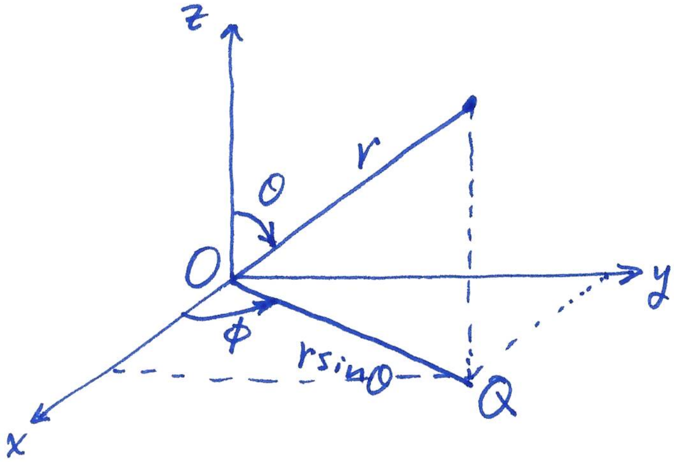
Or $$\boxed{u(x) = \frac{a^2 - |x|^2}{2\pi a}\int_{|x'|=a}\frac{u(x')}{|x-x'|^2}\,ds'}$$



$(ds' = dS_{x'} = a\,d\phi.)$

Observe: $u(r,\theta)$ is completely determined by its values on the boundary $r=a$!

Spherical coordinates



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\phi \end{cases}$$

$(r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi < 2\pi)$

Laplacian $\quad \Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin^2\theta}u_{\phi\phi} + \frac{1}{r^2}u_{\theta\theta} + \frac{\cot\theta}{r^2}u_\theta$

Or. $\Delta u = \frac{1}{r}(ru)_{rr} + \frac{1}{r^2\sin^2\theta}u_{\phi\phi} + \frac{1}{r^2\sin\theta}(\sin\theta\, u_\theta)_\theta$

Or: $\Delta u = \frac{1}{r^2}(r^2u_r)_r + \frac{1}{r^2\sin^2\theta}u_{\phi\phi} + \frac{1}{r^2\sin\theta}(\sin\theta\, u_\theta)_\theta$

Laplace's equation in a sphere

$$\begin{cases} \Delta u = 0, & x^2+y^2+z^2 < a^2 \\ u(x,y,z) = F(\theta), & x^2+y^2+z^2 = a^2 \end{cases}$$

u: indep. of ϕ.

$$\begin{cases} r(ru)_{rr} + \frac{1}{\sin\theta}(\sin\theta\, u_\theta)_\theta = 0 & (r > 0,\ 0 \leq \theta \leq \pi) \\ u(r = a, \theta) = F(\theta). & (0 \leq \theta \leq \pi) \end{cases}$$

$u(r,\theta) = R(r)\,\Theta(\theta).$

$$\frac{1}{\sin\theta\,\Theta(\theta)}(\sin\theta\,\Theta_\theta)_\theta = -\frac{r}{R(r)}(r R(r))_{rr} = -\lambda = \text{const}.$$

$$r(rR)_{rr} - \lambda R = 0 \qquad (0 \le r < a)$$

$$\frac{1}{\sin\theta}(\sin\theta\, \Theta_\theta)_\theta + \lambda \Theta = 0 \qquad (0 \le \theta \le \pi)$$

Let $x = \cos\theta$

$$\sin\theta \frac{d\Theta}{d\theta} = (1-\cos^2\theta)\frac{1}{\sin\theta}\frac{d\Theta}{d\theta} = -(1-x^2)\frac{d\Theta}{dx}$$

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \lambda\Theta = 0 \qquad (-1 < x < 1)$$

This is Legendre's equation.

$$\lambda_n = n(n+1), \quad n = 0, 1, 2, \cdots$$

$$\Theta_n(x) = P_n(x)$$

$$\Theta_n(\cos\theta) = P_n(\cos\theta), \quad n = 0, 1, 2, \cdots$$

P_n: Legendre's polynomials.

$$r^2 R'' + 2rR' - \lambda R = 0 \qquad \lambda = n(n+1) \quad n = 0, 1, 2, \cdots$$

Cauchy-Euler equation

$R = r^\alpha \Rightarrow R = c_1 r^n + c_2 r^{-n-1}$ $\quad \alpha = n, -(n+1)$ $\qquad c_2 = 0$ as $R(0)$ is finite.

$$R_n(r) = r^n \quad (n = 0, 1, 2, \cdots)$$

$$u(r,\theta) = \sum_{n=0}^{\infty} B_n r^n P_n(\cos\theta)$$

$$F(\theta) = u(a,\theta) = \sum_0^\infty B_n a^n P_n(\cos\theta) \qquad (0 < \theta < \pi)$$

Let $f(x) = F(\cos^{-1}x)$:

$$f(x) = \sum_{n=0}^{\infty} A_n P_n(x) \qquad (-1 < x < 1)$$

$\{P_n\}_{n=0}^{\infty}$: orthogonal in $L^2(-1,1)$. $\quad \int_{-1}^{1} P_n P_m\, dx = \begin{cases} 0, & n \ne m \\ \frac{2}{2n+1}, & n = m \end{cases}$

$$\boxed{A_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) P_n(x)\, dx, \quad n = 0, 1, 2, \cdots}$$

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n \left(\frac{r}{c}\right)^n P_n(\cos\theta) \qquad (r \le a)}$$

Exterior problem

$$\begin{cases} \Delta u = 0 & r > a \\ u \text{ bdd at } \infty & \\ u(r=a,\theta) = F(\theta). & \end{cases}$$

Solution.

$$\boxed{\begin{aligned} u(r,\theta) &= \sum_{n=0}^{\infty} A_n \left(\frac{c}{r}\right)^{n+1} P_n(\cos\theta) \\ A_n &= \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx \quad (n=0,1,2,\dots) \\ f(x) &= F(\cos^{-1}x). \quad (-1<x<1) \end{aligned}}$$

Appendix. Legendre Polynomials

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right], \quad n=0,1,2,\dots \quad \left(\begin{array}{c}\text{Rodrigue's} \\ \text{formula}\end{array}\right)$$

$[\ P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}(3x^2-1),\ P_3(x)=\frac{1}{2}(5x^3-3x),$
$P_4(x)=\frac{1}{8}(35x^4-30x^2+3),\ P_5(x)=\frac{1}{8}(63x^5-70x^3+15x),\dots]$.

① Orthogonality: $\displaystyle\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n=m. \end{cases}$

② Even/odd: P_n is even/odd if n is even/odd.

③ P_n $(n\geq 1)$ has exactly n distinct roots in $(-1,1)$.

④ Recurrence relation $\qquad P_0(x)=1,\ P_1(x)=x.$

$$nP_{n-1}(x) - (2n+1)xP_n(x) + (n+1)P_{n+1}(x) = 0 \quad (n=1,2,\dots)$$

⑤ Generating function $\displaystyle\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \quad \left(\begin{array}{c}-1\leq x\leq 1 \\ -1<t<1\end{array}\right)$

⑥ Legendre polynomials as eigenfunctions.

For $y=y(x)$, define $(Ly)(x) = -\left((1-x^2)y'\right)'$

$$\begin{cases} Ly = \lambda y \\ y(-1)=y(1)=0 \end{cases} \Longrightarrow \begin{array}{l} \lambda = n(n+1), \quad n=0,1,2,\dots \\ y = P_n(x). \end{array}$$

Note: ① (together with $P_n(1)=1$ $(n=0,1,\dots)$) or ⑥ can be used to define Legendre polynomials $P_n(x)$.