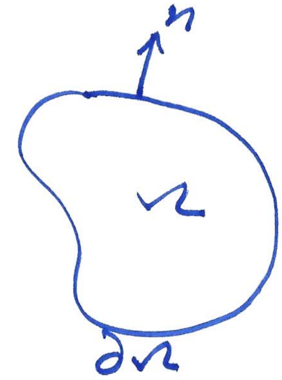
Section 1.2 Fundamental Solutions and Green's functions



Green's identities

(1) $$\int_{\partial\Omega} v \frac{\partial u}{\partial n}\, dS = \int_{\Omega} (v\Delta u + \nabla v\cdot\nabla u)\, dx$$

(2) $$\int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds = \int_{\Omega} (v\Delta u - u\Delta v)\, dx$$

where $u, v \in C^2(\Omega)\cap C^1(\bar{\Omega})$, $\Omega \subset \mathbb{R}^d$: bded and smooth domain, and n is the unit exterior normal to the boundary $\partial\Omega$ of Ω. $x = (x_1, \dots, x_d)$.

$$\frac{\partial u}{\partial n} = \nabla u \cdot n \qquad \nabla u = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{bmatrix} \qquad \vec{n} = \begin{bmatrix} n_1 \\ \vdots \\ n_d \end{bmatrix}$$

<u>Proof</u> Divergence Theorem

$$\int_{\partial\Omega} \vec{V}\cdot\vec{n}\, dS = \int_{\Omega} \nabla\cdot\vec{V}\, dx$$

$$\vec{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_d \end{bmatrix}, \quad \nabla\cdot\vec{V} = \sum_{j=1}^{d} \frac{\partial V_j}{\partial x_j}.$$

Set $\vec{V} = v\nabla u$. Then $\nabla\cdot\vec{V} = \nabla v\cdot\nabla u + v\Delta u$.

$\vec{V}\cdot\vec{n} = v\frac{\partial u}{\partial n}$.

$$\int_{\Omega} \nabla\cdot\vec{V}\, dx = \int_{\Omega} (v\Delta u + \nabla v\cdot\nabla u)\, dx$$
$$= \int_{\partial\Omega} \vec{V}\cdot\vec{n}\, dS = \int_{\partial\Omega} v\frac{\partial u}{\partial n}\, dS.$$

This is (1). Now, switch u and v in (1):

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} dS = \int_{\Omega} (u\Delta v + \nabla u \cdot \nabla v) dx.$$

Subtracting this from (1), we obtain (2). Q.E.D.

Consider now

(3) $$-\Delta K = \delta$$ [K is the electrostatic potential when there is a point charge (unit charge) placed at O.]

$K = K(x)$, $x \in \mathbb{R}^3$, δ = the Dirac δ-function at O.

By symmetry, we assume $K = K(r)$, $r = |x|$.

$$\Delta K = r^{-2}(r^2 K')' = K'' + \frac{2}{r} K'.$$

If $r > 0$, then $\Delta K(x) = 0$. (since $\delta = 0$ at $x \neq 0$).

So, $(r^2 K')' = 0$ $\quad r^2 K' = -c_1$

$K' = -\frac{c_1}{r^2}$ $\quad K(r) = c_2 + \frac{c_1}{r}$.

Note: if K satisfies (3), then $K + c$ (c: const.) also satisfies (3). So, let us set $c_2 = 0$. This also means $K(\infty) = 0$.

$$K(x) = \frac{c_1}{r}.$$

We need to determine c_1. We use (3).

Note that for any smooth function $\phi = \phi(x)$ that vanish outside a ball centered at O,

$$\langle \delta, \phi \rangle = \int_{\mathbb{R}^3} \delta \phi \, dx = \phi(0).$$

The meaning of (3) is $\boxed{\int_{\mathbb{R}^3} \nabla K \cdot \nabla \phi \, dx = \phi(0)}$

or

$$\int_{\mathbb{R}^3} K \Delta \phi \, dx = -\phi(0).$$

[— Integration by parts, two times or: $-\langle \Delta K, \phi \rangle = \langle \delta, \phi \rangle = \phi(0)$

$$\int \Delta K \phi \, dx = -\int \nabla K \cdot \nabla \phi \, dx = \int K \Delta \phi \, dx.$$

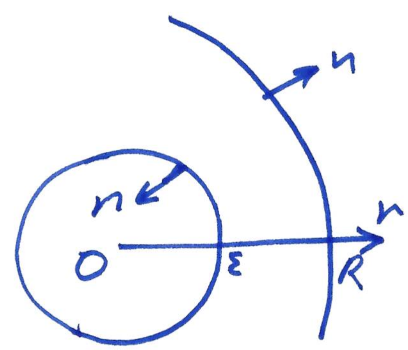
[if K is nice].]

$$\phi(0) = \int_{\mathbb{R}^3} \nabla K \cdot \nabla \phi \, dx$$
$$= \lim_{\varepsilon \to 0^+} \int_{\{r>\varepsilon\}} \nabla K \cdot \nabla \phi \, dx$$

Suppose $\phi \equiv 0$ if $r > R$ for some $R > 0$. Then

$$\int_{\{r>\varepsilon\}} \nabla K \cdot \nabla \phi \, dx = \int_{\{\varepsilon<r<R\}} \nabla K \cdot \nabla \phi \, dx$$

$$= -\int_{\{\varepsilon<r<R\}} \Delta K \, \phi \, dx + \int_{\{r=\varepsilon \text{ or } r=R\}} \phi \frac{\partial K}{\partial n} dS$$



$$\overset{\Delta K = 0 \ (r>0)}{\underset{\phi = 0 \ (r>R)}{=\!=}} \int_{\{r=\varepsilon\}} \phi \frac{\partial K}{\partial n} dS$$

$$\frac{\partial K}{\partial n} = -\frac{d}{dr}(K(r)) = -\frac{d}{dr}\left(\frac{c_1}{r}\right) = c_1 r^{-2}$$

$$\int_{\{r>\varepsilon\}} \nabla K \cdot \nabla \phi \, dx = -c_1 \int_{\{r=\varepsilon\}} \frac{1}{r^2} \phi(x) \, dS$$

$$= -\frac{c_1}{\varepsilon^2} \int_{\{r=\varepsilon\}} \phi(x) \, dS = -\frac{c_1}{\varepsilon^2} \int_{\{r=\varepsilon\}} [\phi(x) - \phi(0)] \, dx$$
$$- \frac{c_1}{\varepsilon^2} \int_{\{r=\varepsilon\}} \phi(0) \, dx$$
$$= -\frac{c_1}{\varepsilon^2} 4\pi\varepsilon^2 O(\varepsilon) - \frac{c_1}{\varepsilon^2} 4\pi\varepsilon^2 \phi(0).$$
$$\longrightarrow c_1 4\pi \phi(0) \quad \text{as } \varepsilon \to 0^+.$$

Thus, $c_1 = \frac{1}{4\pi}$.

$$\boxed{K(x) = \frac{1}{4\pi|x|} \qquad (x \neq 0)}$$

is the fundamental sol'n for $-\Delta$ in $\mathbb{R}^3$

$K(x) = \dfrac{1}{d(d-2)\alpha(d)} \dfrac{1}{|x|^{d-2}}$ $(d \geq 3)$ is the fundamental sol'n for $-\Delta$ in $\mathbb{R}^d$ $(d \geq 3)$. ($\alpha(d)$ = vol. of unit ball in $\mathbb{R}^d$.)

Similarly, $\boxed{K(x) = -\frac{1}{2\pi}\log|x| \qquad (x\neq 0, x\in\mathbb{R}^2)}$ is the fundamental solution for $-\Delta$ in $\mathbb{R}^2$.

In this section, we always denote $d = 2$ or 3, and $K = K(x)$ the corresponding fundamental solution.

<u>Theorem</u>. Let $f \in C(\mathbb{R}^d)$ and define

$$u(x) = \int_{\mathbb{R}^d} K(x-y) f(y)\, dy, \quad x \in \mathbb{R}^d.$$

Then $u \in C^2(\mathbb{R}^d)$ and $-\Delta u = f$ in $\mathbb{R}^d$.

Heuristically,

$$-\Delta u(x) = \int_{\mathbb{R}^d} -\Delta_x K(x-y) f(y)\, dy$$

$$= \int_{\mathbb{R}^d} \delta_x(y) f(y)\, dy = f(x).$$

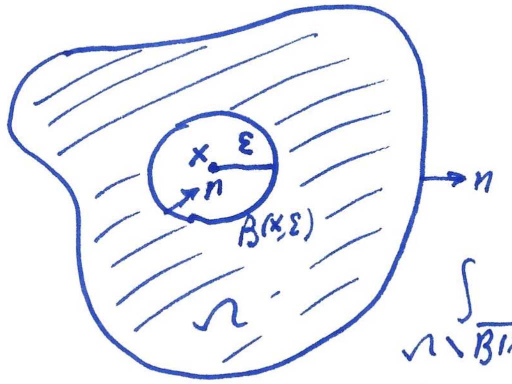
For a proof, see Evans PDE book (Section 2.2).

<u>Theorem</u> If $\Omega \subseteq \mathbb{R}^d$ is smooth and bounded and $u \in C^2(\overline{\Omega})$, then

$$u(x) = -\int_{\Omega} K(y-x)\Delta u(y)\, dy + \int_{\partial\Omega}\left[K(y-x)\frac{\partial u(y)}{\partial n_y} - u(y)\frac{\partial K(y-x)}{\partial n_y}\right] dS_y, \quad \forall x \in \Omega.$$

[Here $\partial\Omega$ denotes the boundary of Ω, and n or n_y denote the unit exterior normal at $y \in \partial\Omega$.]

Proof   Fix $x \in \Omega$.



$$\int_{\Omega} K(y-x)\,\Delta_y u(y)\,dy = \lim_{\varepsilon\to 0^+} \int_{\Omega\setminus\overline{B(x,\varepsilon)}} K(y-x)\,\Delta_y u(y)\,dy$$

$$\int_{\Omega\setminus\overline{B(x,\varepsilon)}} K(y-x)\,\Delta_y u(y)\,dy$$

$$= \int_{\Omega\setminus\overline{B(x,\varepsilon)}} \Big[K(y-x)\,\Delta_y u(y) - u(y)\underbrace{\Delta_y K(y-x)}_{\substack{=0\\ \text{in } \Omega\setminus\overline{B(x,\varepsilon)}}}\Big]\,dy$$

use 2nd Green's identity

$$= \int_{\partial(\Omega\setminus\overline{B(x,\varepsilon)})} \Big[K(y-x)\frac{\partial u(y)}{\partial n_y} - u(y)\frac{\partial K(y-x)}{\partial n_y}\Big]\,dS_y$$

$$= \int_{\partial\Omega} \Big[K(y-x)\frac{\partial u(y)}{\partial n_y} - u(y)\frac{\partial K(y-x)}{\partial n_y}\Big]\,dS_y$$

$$+ \int_{\{y:\,|y-x|=\varepsilon\}} \Big[-\frac{1}{4\pi\varepsilon}\frac{\partial u(y)}{\partial n_y} - u(y)\frac{\partial}{\partial n_y}\Big(-\frac{1}{4\pi|y-x|}\Big)\Big]\,dS_y$$

~~Since n is opposite~~ to $r = y-x$.

$$\int_{\{y:\,|y-x|=\varepsilon\}} \frac{1}{\varepsilon}\frac{\partial u(y)}{\partial n_y}\,dS_y = \frac{1}{\varepsilon}\,O(1)\,4\pi\varepsilon^2 \to 0 \text{ as } \varepsilon\to 0.$$

$$-\int_{\{y:\,|y-x|=\varepsilon\}} u(y)\frac{\partial}{\partial n_y}\frac{1}{4\pi|y-x|}\,dS_y \overset{\substack{n_y=-r/|r|\\ r=y-x}}{=\!=} -\int_{\{y:\,|y-x|=\varepsilon\}} u(y)\cdot\frac{1}{4\pi|y-x|^2}\,dS_y$$

$$= \underbrace{\int_{\{y:\,|y-x|=\varepsilon\}} \frac{1}{4\pi\varepsilon^2}\big[-u(y)+u(x)\big]\,dS_y}_{\substack{\to 0 \text{ as } \varepsilon\to 0\\ \text{since } u \text{ is continuous at } x.}} - \underbrace{\int_{\{y:\,|y-x|=\varepsilon\}} \frac{1}{4\pi\varepsilon^2}u(x)\,dS_y}_{\cancel{=u(x)}\ \ = -u(x).}$$

Q.E.D.

Consider solving $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$

The formula

$$u(x) = -\int_\Omega K(y-x)\overbrace{\Delta u(y)}^{=-f(y)}dy + \int_{\partial\Omega}\Big[K(y-x)\frac{\partial u(y)}{\partial n} - \underbrace{u(y)}_{=g(y)}\frac{\partial K(y-x)}{\partial n}\Big]dS_y \qquad (1)$$

does not represent the solution, really as we need to know $\frac{\partial u}{\partial y}$ on $\partial\Omega$. The idea is to correct $K(y-x)$ So that the corresponding term will vanish.

$\forall x \in \Omega$, define $\phi^x = \phi(y)$ $(y \in \Omega)$ by

$$\begin{cases} \Delta_y \phi^x(y) = 0 & \forall y \in \Omega \\ \phi^x(y) = K(y-x) & \forall y \in \partial\Omega \end{cases}$$

As in Theorem above, we have

$$-\int_\Omega \phi^x(y)\Delta u(y)dy = \int_{\partial\Omega}\Big[u(y)\frac{\partial\phi^x(y)}{\partial n} - \phi^x(y)\frac{\partial u(y)}{\partial n}\Big]dS(y)$$
$$= \int_{\partial\Omega}\Big[u(y)\frac{\partial\phi^x}{\partial n}(y) - K(y-x)\frac{\partial u(y)}{\partial n}\Big]dS(y) \qquad (2)$$

Equations (1) and (2) lead to

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial n}(x,y)dS_y - \int_\Omega G(x,y)\Delta u(y)dy$$

where

$$G(x,y) = K(y-x) - \phi^x(y) \quad (x, y \in \Omega, x \neq y)$$

This is called <u>the Green's function</u> for $-\Delta$ on Ω with the Dirichlet boundary condition.

<u>Theorem</u> If $u \in C^2(\bar\Omega)$ solves $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$

then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial n}dS(y) + \int_\Omega f(y)G(x,y)dy. \quad (x \in \Omega).$$

Property of $G(x,y)$: $G(x,y) = G(y,x)$.

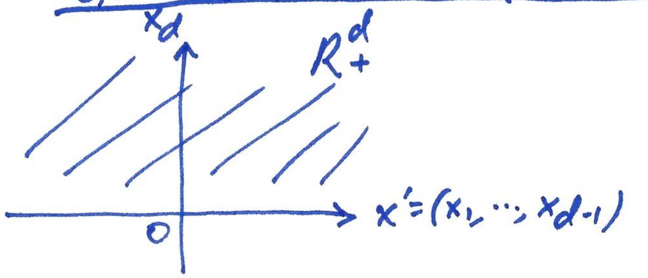
See Thm 13 on p. 35 of Evans' book. (§2.2).

Note that the Green's function $G(x,y)$ regarded as a function of y: $G(x,\cdot)$, is determined by

$$\begin{cases} -\Delta G = \delta_x & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega \end{cases}$$

where δ_x is the Dirac δ-function concentrated at x.

Green's function for the half-space $\mathbb{R}^d_+ = \{(x_1,\cdots,x_d)\in\mathbb{R}^d : x_d > 0\}$



$$\boxed{G(x,y) = K(y-x) - K(y-\tilde{x})}$$

where
$\tilde{x} = (x_1,\cdots,x_{d-1},-x_d)$,
if $x = (x_1,\cdots,x_{d-1},x_d)$.

For example, in $\mathbb{R}^+_3 = \{(x,y,z)\in\mathbb{R}^3 : z>0\}$.

$$G((x,y,z),(x',y',z')) = \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}.$$

Back to $G(x,y) = K(y-x) - K(y-\tilde{x})$

$$\frac{\partial G(x,y)}{\partial n_y} = -\frac{\partial G}{\partial y_d}(x,y) = -\frac{x_d}{2\pi}\cdot\frac{1}{|x-y|^3}$$

Solution to $\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d_+ \\ u = g & \text{on } \partial\mathbb{R}^d_+ \end{cases}$ is (Poisson's formula)

$$\boxed{u(x) = \frac{2x_d}{d\,\alpha(d)} \int_{\partial\mathbb{R}^d_+} \frac{g(y)}{|x-y|^d}\, dy \quad (x \in \mathbb{R}^d_+)}$$

If $d=3$, then

$$u(x) = \frac{x_3}{2\pi} \int_{\{x_3=0\}} \frac{g(y)}{|x-y|^d}\, dy.$$

Green's function for ~~a~~ the unit ball in $\mathbb{R}^d$ (only consider $d=3$)

$$\boxed{\begin{gathered} G(x,y) = K(y-x) - K\big(|x|(y-\tilde{x})\big), \quad \begin{pmatrix} x, y \in B(0,1) \\ x \neq y \end{pmatrix} \\ x = (x_1, x_2, x_3), \quad \cancel{\tilde{x} = (x_1, x_2, -x_3)}. \\ \tilde{x} = \frac{x}{|x|^2} \end{gathered}}$$

Green's function for the ball $B(0,R)$ in $\mathbb{R}^3$

$$\boxed{G(x,y) = K(y-x) - \frac{R}{|x|} K(y - x^*) \quad x, y \in B(0,R), x \neq y}$$

$x^* = \frac{R^2 x}{|x|^2}$ is outside $B(0,R)$ is $x \in B(0,R)$.

Solution to $\begin{cases} \Delta u = 0 & \text{in } B(0,R) \subseteq \mathbb{R}^3 \\ u = g & \text{on } \partial B(0,R) \end{cases}$ is

$$u(x) = \frac{R^2 - |x|^2}{4\pi R} \int_{\partial B(0,R)} \frac{g(y)}{|x-y|^3}\, dS(y). \quad (x \in B(0,R))$$