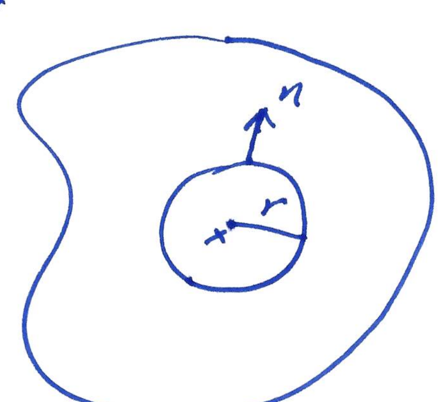
Section 1.3 Mean-Value Theorem / Maximum Principles / Well Posedness

We describe some basic properties of solutions to Poisson's and Laplace's equations, (without solving these equations explicitly). These include the mean-value property of harmonic functions (functions u such that $\Delta u = 0$), the maximum principle, and solution regularity. We shall use these properties to prove the existence, uniqueness, and stability of solutions. We will also introduce the energy method.

Mean-Value Theorem for harmonic functions. Let Ω be an open set in $\mathbb{R}^d$. Suppose $\Delta u = 0$ in Ω. Then

$$u(x) = \fint_{\partial B(x,r)} u\,dS = \fint_{B(x,r)} u(y)\,dy$$

for each ball $B(x,r) \subset \Omega$



Ω

Notation

$$B(x,r) = \{ y \in \mathbb{R}^d : |y-x| < r \}.$$

$$\partial B(x,r) = \text{boundary of } B(x,r) = \{ y \in \mathbb{R}^d : |y-x| = r \}$$

$\fint$: denotes the average.

e.g., $\displaystyle \fint_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$\fint_\Omega g(x)\,dx = \frac{1}{|\Omega|}\int_\Omega g(x)\,dx.$$

In $\mathbb{R}^3$, $$⨍_{\partial B(x,r)} u\, ds = \frac{1}{4\pi r^2} \int_{\{y \in \mathbb{R}^3 : |y-x|=r\}} u(y)\, dy$$

$4\pi r^2 = |\partial B(x,r)| =$ surface area of ~~sphere~~ $\partial B(x,r)$.

Proof

$$\phi(r) := ⨍_{\partial B(x,r)} u(y)\, dS(y) = \frac{1}{|\partial B(x,r)|} \int_{\{y : |y-x|=r\}} u(y)\, dS(y)$$

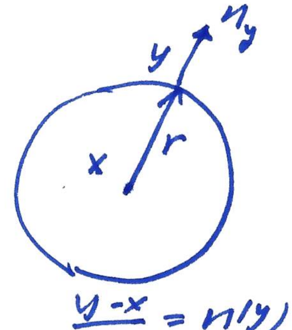
$$\overset{y=x+rz}{=} \frac{1}{r^{d-1}|\partial B(0,1)|} \int_{\{z : |z|=1\}} u(x+rz)\, r^{d-1}\, dS(z)$$

$$= \frac{1}{|\partial B(0,1)|} \int_{|z|=1} u(x+rz)\, dS(z).$$

$$\phi'(r) = \frac{1}{|\partial B(0,1)|} \int_{|z|=1} \nabla u(x+rz) \cdot z\, dS(z)$$

$$\overset{z=\frac{y-x}{r}}{=} \frac{1}{|\partial B(x,r)|} \int_{\{y : |y-x|=r\}} \nabla u(y) \cdot \frac{y-x}{r}\, dS(y)$$

$$= \frac{1}{|\partial B(x,r)|} \int_{\{y : |y-x|=r\}} \frac{\partial u(y)}{\partial n_y}\, dS(y)$$



$\frac{y-x}{r} = n(y)$

Green's formula or Gauss' formula
$$= \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y)\, dy \cancel{S(y)}$$

$$= 0$$

$\phi(r) = \text{const.}$ for $r \in (0, r_0)$, where $B(x, r_0) \subset \Omega$.

$$\phi(r) = \lim_{t \to 0} \phi(t) = \lim_{t \to 0} ⨍_{B(x,t)} u(y)\, dS(y) = u(x).$$

$$\int_{B(x,r)} u(y)\,dy = \int_0^r \left(\int_{\partial B(x,s)} u\,dS \right) ds$$

$$= \int_0^r u(x)\, |\partial B(0,1)|\, s^{d-1}\, ds$$

$$= u(x) \int_0^r |\partial B(x,s)|\, ds$$

$$= u(x)\, |B(x,r)|.$$

Q.E.D.

Theorem (Converse to mean-value property). If $\Omega \subseteq \mathbb{R}^d$ is open, $u \in C^2(\Omega)$ satisfies

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y)$$

for any ball $B(x,r) \subset \Omega$, then u is harmonic in Ω.

Proof. If $\Delta u \not\equiv 0$, then there exists a ball $B(x,r) \subset \Omega$, say, $\Delta u > 0$ in $B(x,r)$. Now, set as above

$$\phi(r) = \fint_{\partial B(x,r)} u(y)\, dS(y).$$

By assumption, $\phi(r) = u(x)$ indep. of r. So $\phi'(r) = 0$. On the other hand, by the same calculation above:

$$\phi'(r) = \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y)\, dy > 0.$$

A contradiction. Q.E.D.

Question. For 2-dimensional space, can we replace a disk by a square for mean-value property?
[Answer: No. Generally not true for any $d \geq 2$.]

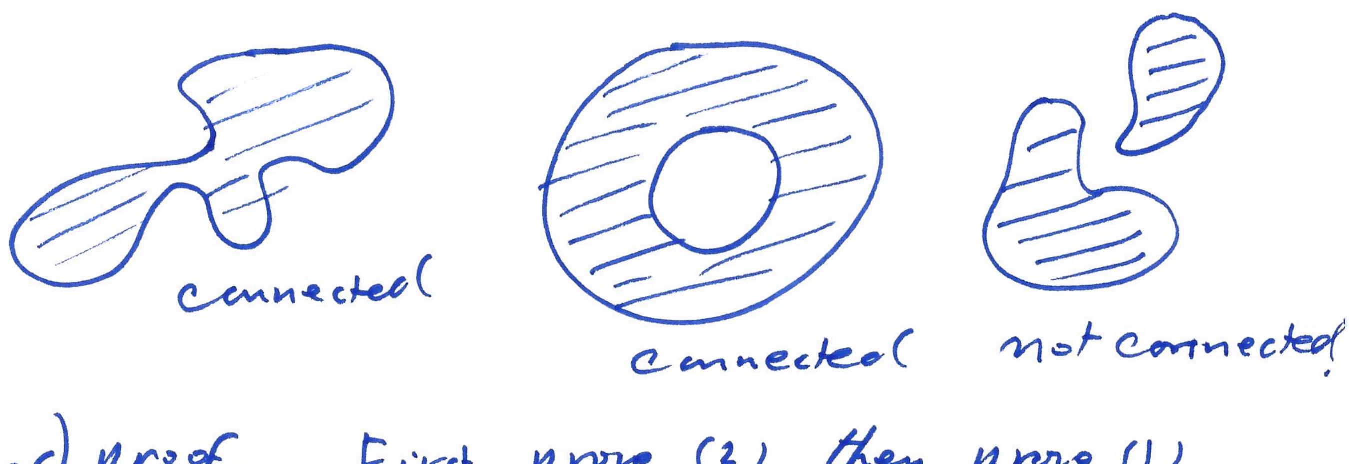
<u>Theorem (Strong maximum principle)</u>. Let Ω be an open set in $\mathbb{R}^d$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in Ω.

(1) Then $$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u,$$

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

[i.e., the maximum (or minimum) value of a harmonic function is attained at boundary.]

(2) Suppose further that Ω is connected, and there exists $x_0 \in \Omega$ such that
$$u(x_0) = \max_{x\in\overline{\Omega}} u(x),$$
$$\left[\text{or} \quad u(x_0) = \min_{x\in\overline{\Omega}} u(x). \right]$$
then $u = \text{const.}$ in Ω.



(Idea of) <u>proof</u>. First, prove (2), then prove (1).

Say $u(x_0) = M = \max_{\overline{\Omega}} u(x)$. $B(x_0, r) \subset \Omega$.

Then $u(x_0) = \fint_{B(x_0,s)} u(y)\,dy \le M = u(x_0)$. So, $u \equiv M$ in $B(x_0, s)$ $\forall s \le r$. Some geometrical argument using the connectness $\Longrightarrow$ $u \equiv M$ on Ω. <u>Q.E.D.</u>

Corollary Let Ω be a connected open set in $\mathbb{R}^d$. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \Omega . \\ u = g & \text{on } \partial\Omega , \end{cases}$$

~~Suppose~~ for some $g \geq 0$ on $\partial\Omega$. Then $u \geq 0$ in Ω. If $g > 0$ at one pt on $\partial\Omega$, then $u > 0$ in Ω.

Q.E.D.

Similarly. $\begin{cases} \Delta u = 0 & \Omega \\ u = g & \partial\Omega \end{cases}$ $\quad g \underset{\partial\Omega}{\leq} 0 \Longrightarrow u \leq 0$ in Ω.

$g < 0$ at one pt on $\partial\Omega$ $\Longrightarrow u < 0$ in Ω.

Theorem (Uniqueness of solution to Dirichlet boundary-value problem of Poisson's equation) Let $\Omega \subset \mathbb{R}^d$ be open. $f \in C(\Omega)$, $g \in C(\partial\Omega)$. Then, there exists at most one solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega , \\ u = g & \text{on } \partial\Omega . \end{cases}$$

Proof If both u_1, u_2 solve the same problem

$$\begin{cases} -\Delta u_1 = f & \Omega , \\ u_1 = g & \partial\Omega , \end{cases} \quad \begin{cases} -\Delta u_2 = f & \Omega , \\ u_2 = g & \partial\Omega , \end{cases}$$

then $u = u_1 - u_2$ solves $\begin{cases} \Delta u = 0 & \Omega . \\ u = 0 & \partial\Omega . \end{cases}$

Hence $u = 0$ in Ω. Q.E.D.

Theorem (Regularity) If u is harmonic in an open set in $\mathbb{R}^d$, then $u \in C^\infty(\Omega)$. In fact, u is analytic in Ω (i.e., u is a power series in Ω).

[See L.C. Evans, PDE, AMS, 2010, p. 31.]

We now introduce the energy method.

Consider
$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Assume, e.g.
$f \in C(\bar{\Omega})$
$g \in C(\partial\Omega)$

Here we assume $\Omega \subset \mathbb{R}^d$ is a bounded and smooth domain (a domain is a connected open set.). We define

$$I[v] = \int_\Omega \left[\frac{1}{2}|\nabla v(x)|^2 - f(x)v(x)\right] dx$$

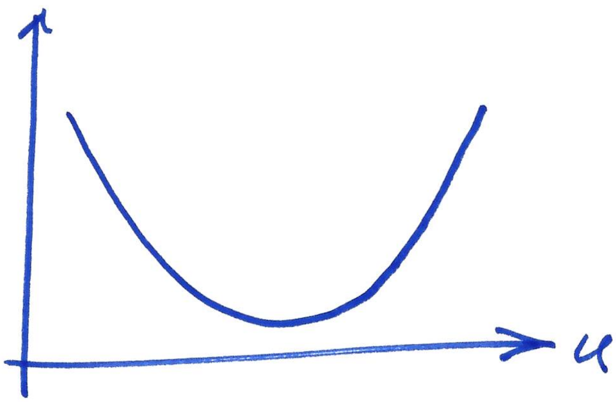
Here, $v : \Omega \to \mathbb{R}$ is a function. We may assume v is smooth in Ω. say, $v \in C^1(\bar{\Omega})$. We say $I : C^1(\bar{\Omega}) \to \mathbb{R}$ is a functional. It represents some potential energy. For instance, it can be the elastic energy. [Note: for electrostatic energy, the sign is a tricky issue.]

Observe that $I[\cdot]$ is a convex functional.

i.e., $$I[\lambda_1 v_1 + \lambda_2 v_2] \le \lambda_1 I[v_1] + \lambda_2 I[v_2]$$

for any $v_1, v_2 \in C^1(\bar{\Omega})$ and $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 + \lambda_2 = 1$.

Equivalently, $\delta^2 I[v] \geq 0$ for any $v \in C^1(\bar{\Omega})$, where $\delta^2 I$ is the second-order variational derivative, or second variation.



Likely, $I[\cdot]$ attains its minimum value at some u. This is in fact true: there exists $u \in C^1(\bar{\Omega})$ s.t.

$$I[u] \leq I[v] \quad \forall v \in C^1(\bar{\Omega}).$$

Now, consider any perturbation $\varphi = \varphi(x)$ of u. Assume $\varphi \in C^1(\bar{\Omega})$ but $\varphi = 0$ on $\partial\Omega$. (In general, consider $\varphi \in C_c^1(\bar{\Omega})$, where the subscript c denotes compact support. $\varphi \in C_c^1(\bar{\Omega})$ means $\varphi = 0$ close to boundary $\partial\Omega$.)

We have

$$I[u] \leq I[u + t\varphi] \quad \forall t \in \mathbb{R}.$$

Let $g(t) = I[u + t\varphi]$.

Then $g(0) \leq g(t) \quad \forall t$.

So, $g'(0) = 0$.

$$0 = g'(0) = \frac{d}{dt}\Big|_{t=0} I[u + t\varphi]$$

$$= \frac{d}{dt}\Big|_{t=0} \int_\Omega \left[\frac{1}{2}|\nabla u + t\nabla\varphi|^2 - f(u + t\varphi)\right] dx$$

$$= \frac{d}{dt}\Big|_{t=0} \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + t\,\nabla u\cdot\nabla\varphi + t^2|\nabla\varphi|^2 - fu - tf\varphi\right]dx$$

$$= \int_\Omega \left[\nabla u\cdot\nabla\varphi + 2t\,|\nabla\varphi|^2 - f\varphi\right]dx\Big|_{t=0}$$

$$= \int_\Omega (\nabla u\cdot\nabla\varphi - f\varphi)\,dx$$

Suppose $u \in C^2(\bar\Omega)$

$$= \int_\Omega (-\Delta u\cdot\varphi - f\varphi)\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}\varphi\,dS$$

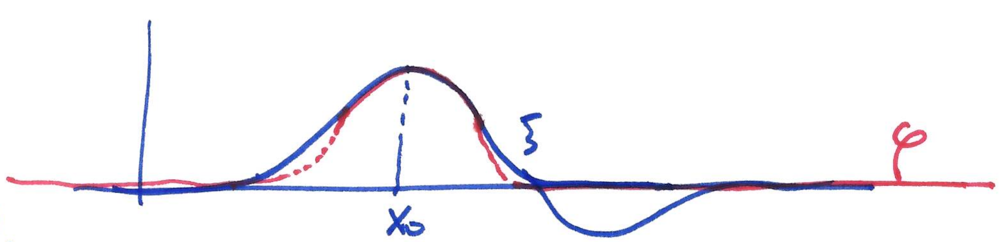
$$= \int_\Omega (-\Delta u - f)\,\varphi\,dx.$$

Hence. if $u \in C^2(\bar\Omega)$, and $I[u] \le I[v]$ $\forall v$.
then $\int_\Omega (\Delta u + f)\,\varphi\,dx = 0 \quad \forall \varphi \in C_c^1(\bar\Omega)$.

<u>Lemma of variations</u> Let Ω be open in $\mathbb{R}^d$.
Suppose $\xi = \xi(x) \in C(\Omega)$ satisfies

$$\int_\Omega \xi(x)\,\varphi(x)\,dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

then $\xi(x) \equiv 0 \quad \forall x \in \Omega$. <u>Q.E.D.</u>



In 1-d:
If $\xi(x_0) > 0$, then construct φ s.t. $\int \xi\varphi > 0$
to get a contradiction. <u>Q.E.D.</u>

<u>Conclusion</u> If $u \in C^2(\bar{\Omega})$ satisfies $u = g$ on $\partial\Omega$ and $I[u] \le I[v]$ for any $v \in C^1(\bar{\Omega})$ satisfying $v = g$, then $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$

We call $-\Delta u = f$ <u>the Euler-Lagrange equation</u> for $I[u] = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - fu\right] dx$.

Define $\mathcal{A} = \{v \in C^2(\bar{\Omega}) : v = g \text{ on } \partial\Omega\}$.

<u>Theorem</u> (<u>Dirichlet's principle</u>) Assume $u \in C^2(\bar{\Omega})$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (*)$$

Then

$$I[u] = \min_{v \in \mathcal{A}} I[v]. \quad (**)$$

Conversely, if $u \in \mathcal{A}$ solves $(**)$, then it solves $(*)$.

<u>Proof</u>. $(**)$ is equivalent to

$$I[u] \le I[u + t\varphi] \quad \forall t \in \mathbb{R}, \forall v:$$

$v \in C^2(\bar{\Omega})$, $v = 0$ on $\partial\Omega$.

The latter implies $g'(0) = 0$ with $g(t) = I[u + t\varphi]$.

As above, this implies $(*)$

Now, prove $(*) \Longrightarrow (**)$.

$\forall w \in \mathcal{A}$: $0 = \int_\Omega (-\Delta u - f)(u - w)\, dx$

$$= \int_\Omega [\nabla u \cdot \nabla(u-w) - f(u-w)]\, dx$$

Note: $u - w = g - g = 0$ on $\partial\Omega$

Hence, $\int_\Omega \left(|\nabla u|^2 - uf\right) dx = \int_\Omega (\nabla u \cdot \nabla w - wf)\, dx$

$$\le \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2 - wf\right) dx,$$

where we used the Cauchy-Schwarz inequality:

$$|\vec{a}\cdot\vec{b}| \leq |\vec{a}|\,|\vec{b}| \leq \frac{1}{2}|\vec{a}|^2 + \frac{1}{2}|\vec{b}|^2$$

with $\vec{a} = \nabla u$ and $\vec{b} = \nabla w$. Finally, rearranging terms to get $I[u] \leq I[w]$. Q.E.D.

Theorem (Uniqueness) There exists at most one solution $u \in C^2(\bar{\Omega})$ to $\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$

Proof. If both u_1, u_2 are solutions, then $u = u_1 - u_2$ solves $\begin{cases} -\Delta u = 0 & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega. \end{cases}$ Hence

$$0 = -\int_\Omega u\,\Delta u\,dx = \int_\Omega |\nabla u|^2 dx$$

So, $\nabla u \equiv 0$ in Ω. $u = \text{const}$ (in each connected component).
But $u = 0$ on $\partial\Omega$, so. $u \equiv 0$ in Ω. Q.E.D.

This method can be generalized. For instance, the fact that $\begin{cases} -\Delta u + \kappa^2 u = f & \text{in } \Omega \\ \partial_n u = g & \text{on } \partial\Omega \end{cases}$ $(\kappa = \text{const.} > 0)$

has at most one solution can be proved similarly.

Let u_1, u_2 solve this prob. then $u = u_1 - u_2$ solves $-\Delta u + \kappa^2 u = 0$, Ω $\quad \partial_n u = 0$ on $\partial\Omega$. Hence

$$0 = \int_\Omega u(-\Delta u + \kappa^2 u)\,dx = \int_\Omega (|\nabla u|^2 + \kappa^2 u^2)\,dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} u\,dS$$

$$= \int_\Omega (|\nabla u|^2 + \kappa^2 u^2)\,dx. \text{ So, } |\nabla u|^2 + \kappa^2 u^2 \equiv 0 \text{ in } \Omega.$$

Hence $u = 0$ in Ω. Q.E.D.

A remark on the solvability and uniqueness for
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases} \qquad (*)$$

By the divergence theorem
$$\int_{\partial\Omega} g\,dS = \int_{\partial\Omega} \frac{\partial u}{\partial n}\,dS = \int_{\Omega} \Delta u\,dS = -\int_{\Omega} f\,dx$$

So,
$$\boxed{\int_{\Omega} f\,dx + \int_{\partial\Omega} g\,dS = 0}$$
This is the solvability condition for $(*)$.

Note that the solution to $(*)$ is in general not unique: if u solves $(*)$ then $u+c$, for any constant c, also solves $(*)$.

<u>Proposition</u> If $u_1, u_2 \in C^2(\bar{\Omega})$ solve $(*)$ then $u_1 - u_2 = \text{const}$ in Ω (in each connected component of Ω).

<u>Pf</u> $u \equiv u_1 - u_2$ solves $\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \partial_n u = 0 & \text{on } \partial\Omega. \end{cases}$

Hence
$$0 = \int_{\Omega} -u\Delta u\,dx = \int_{\Omega} |\nabla u|^2 dx - \int_{\partial\Omega} u \frac{\partial u}{\partial n}\,dS = \int_{\Omega} |\nabla u|^2 dx.$$

Hence, $\nabla u \equiv 0$ in Ω. $u = \text{const.}$ in Ω. <u>Q.E.D.</u>