Chapter 2 The Heat Equation

Section 2.1 Method of separation of variables / Eigenfunction Expansion
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Section 2.1 Method of Separation of Variables

$u = u(x,t)$: $0 < x < L$, $t > 0$. $D = \text{const.} > 0$. $f(x)$: given.

Solve
$$\begin{cases} u_t = D u_{xx} & (0 < x < L,\ t > 0) \\ u(0,t) = 0,\ u(L,t) = 0 & (t > 0) \\ u(x,0) = f(x) & (0 < x < L) \end{cases}$$

Try $u(x,t) = \phi(x) G(t)$

$$\phi G' = D \phi'' G. \qquad \frac{G'(t)}{D G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda(x,t).$$

$\lambda = \text{const.}$ $\qquad G' = -D\lambda G \Longrightarrow G(t) = C e^{-\lambda D t}$.
C: const.

$$\begin{cases} \phi''(x) + \lambda \phi(x) = 0 \\ \phi(0) = 0,\ \phi(L) = 0 \end{cases} \qquad [\text{since } u(0,t) = 0,\ u(L,t) = 0].$$

An eigenvalue problem for $L = -\frac{d^2}{dx^2}$.

(1) $\lambda > 0$. $\phi(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$

$\phi(0) = 0 \Longrightarrow C_1 = 0$. $\phi(L) = 0 \Longrightarrow \sin(\sqrt{\lambda} L) = 0$

$\lambda = \left(\frac{n\pi}{L}\right)^2$ $(n = 1, 2, \cdots)$

$\phi(x) = \sin\frac{n\pi x}{L}$ $(n = 1, 2, \cdots)$

(2) $\lambda = 0$. $\phi''(x) = 0$. $\phi(x) = C_1 + C_2 x$.

$\phi(0) = 0$, $\phi(L) = 0 \Longrightarrow C_1 = C_2 = 0$. $\phi \equiv 0$.

(3) $\lambda < 0$. Let $\mu = -\lambda > 0$, $\phi'' - \mu\phi = 0$

$\phi(x) = C_1 e^{\sqrt{\mu}x} + C_2 e^{-\sqrt{\mu}x}$

$\phi(0) = 0$, $\phi(L) = 0 \Longrightarrow C_1 = C_2 = 0$. $\phi \equiv 0$.

$$u_n = u_n(x,t) = C_n e^{-\lambda_n D t} \phi_n(x)$$
$$= C_n e^{-\left(\frac{n\pi}{L}\right)^2 D t} \sin\frac{n\pi x}{L} \quad (n = 1, 2, \cdots)$$

Principle of superposition

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{L}\right)^2 D t} \sin\frac{n\pi x}{L}.$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}. \qquad x \in (0, L).$$

$\left\{\sin\frac{n\pi x}{L}\right\}_{n=1}^{\infty}$ is an orthogonal system in $L^2(0,L)$.

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n. \end{cases}$$

So. $$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = \sum_{n=1}^{\infty} C_n \int_0^L \sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L}\,dx$$
$$= C_m \cdot \frac{L}{2}.$$

$$\boxed{\begin{aligned} C_m &= \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx, \quad m = 1, 2, 3, \cdots \\ u(x,t) &= \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n\pi}{L}\right)^2 D t} \sin\frac{n\pi x}{L}. \end{aligned}}$$

Formal calculation: $\lim_{t\to+\infty} u(x,t) = 0$.

since $e^{-\left(\frac{n\pi}{L}\right)^2 D t} \to 0$ as $t \to +\infty$.

Steady-state
$-u_{xx} = 0$
$u(0) = 0$, $u(L) = 0$
$\Longrightarrow u(x) \equiv 0$.

Similar for other boundary conditions.

$$\begin{cases} u_t = D u_{xx} & (0<x<L,\ t>0) \\ u_x(0,t)=0,\quad u_x(L,t)=0 & (t>0) \\ u(x,0)=f(x) & (0<x<L) \end{cases}$$

$u(x,t) = \phi(x)\, G(t)$. $\qquad G(t) = c\, e^{-D\lambda t}$

$$\begin{cases} \phi''(x) + \lambda \phi(x) = 0 & (0<x<L) \\ \phi'(0) = \phi'(L) = 0 \end{cases}$$

(1) $\lambda > 0$: $\quad \lambda = \left(\frac{n\pi}{L}\right)^2$, $\quad \phi(x) = C_n \cos\frac{n\pi x}{L}$, $\quad G(t) = c\, e^{-D\left(\frac{n\pi}{L}\right)^2 t}$ $\quad (n=1,2,3,\dots)$

(2) $\lambda = 0$: $\quad \phi(x) = C_0 \qquad G(t) = \text{const}$.

(3) $\lambda < 0$: $\quad$ no solution of ϕ.

Superposition:

$$\boxed{\begin{array}{l} u(x,t) = C_0 + \displaystyle\sum_{n=1}^{\infty} C_n e^{-(n\pi/L)^2 D t} \cos\frac{n\pi x}{L}. \\ C_0 = \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx, \quad C_n = \dfrac{2}{L}\displaystyle\int_0^L f(x) \cos\frac{n\pi x}{L}\,dx, \\ (n=1,2,3,\dots) \end{array}}$$

Fact: $$\int_0^L \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0. \end{cases}$$

Note that $\lim_{t\to\infty} u(x,t) = C_0 = \frac{1}{L}\int_0^L f(x)\,dx$.

This limit is a constant function. It solves the steady-state problem: $-u_{xx} = 0$, $u'(0) = u'(L) = 0$. Any constant function solves this prob. But why the limit (as $t\to\infty$) is $\frac{1}{L}\int_0^L f(x)\,dx$?

Reason: the total temperature is conserved.

$u_t = u_{xx} \implies \int_0^L u_t\,dx = \int_0^L u_{xx}\,dx = u_x\Big|_0^L = u_x(L) - u_x(0) = 0$

$\implies \frac{d}{dt}\int_0^L u(x,t)\,dx = 0 \implies \int_0^L u(x,t)\,dx = \text{const}.$

In particular,

$$\lim_{t\to\infty}\int_0^L u(x,t)\,dx = \int_0^L u(x,0)\,dx = \int_0^L f(x)\,dx$$

$$\int_0^L \lim_{t\to\infty} u(x,t)\,dx = \int_0^L c_0\,dx = c_0\cdot L.$$

So, $c_0 = \frac{1}{L}\int_0^L f(x)\,dx.$

Finally consider periodic boundary conditions.

$$\begin{cases} u_t = D u_{xx} & \text{in } R\times(0,\infty) \\ u(x,t) \text{ is } 2L\text{-periodic in } x & \text{in } t\in(0,\infty) \\ u(x,0) = f(x). \end{cases}$$

$u(x,t) = \phi(x)G(t),\qquad G(t) = Ce^{-D\lambda t}$

$$\begin{cases} \phi''(x) + \lambda\phi(x) = 0 & (0<x<L) \\ \phi(\cancel{0}\,{-L}) = \phi(L)\cancel{=0}, & \phi'(\cancel{0}\,{-L}) = \phi'(L). \end{cases}$$

$\leftarrow$ periodic conditions give ~~us~~ us more. But these are enough to determine $\phi(x)$.

(1) $\lambda>0.\quad \lambda = \left(\frac{n\pi}{L}\right)^2\ (n=1,2\cdots)$

$\phi(x) = \cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L},\ (n=1,2\cdots)$

(2) $\lambda=0.\quad \phi(x) = c_0.$

(3) $\lambda<0\quad$ no solution to $\phi(x)$

$$\boxed{\begin{aligned} u(x,t) &= a_0 + \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2 Dt}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right). \\ a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad \left[\begin{matrix} a_n \\ b_n \end{matrix} = \frac{1}{L}\int_{-L}^{L} \begin{matrix}\cos\frac{n\pi x}{L} \\ \sin\frac{n\pi x}{L}\end{matrix}\, dx\right] \\ a_n &= \frac{1}{L}\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx \\ &(n=1,2,\cdots) \end{aligned}}$$

Fact: $\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n\neq m \\ L & n=m\neq 0 \\ 2L & n=m=0 \end{cases}\qquad \int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n\neq m \\ L & n=m\neq 0 \end{cases}$

$\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0.$

Summary of eigen-value problems: $\phi'' + \lambda\phi = 0$

B.C.	$\phi(0)=0,\ \phi(L)=0$	$\phi'(0)=\phi'(L)=0$	$(-L,L)$ or $(0,2L)$ periodic
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n=1,2,3,\cdots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,1,2,\cdots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,1,2,3,\cdots$
Eigen functions $\phi_n=\phi_n(x)$	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ $(n=1,2,\cdots)$ $\cos\frac{n\pi x}{L}$ $(n=0,1,2,\cdots)$
series	$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$	$f(x)=\frac{A_0}{2}+\sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}$	$f(x)=\boxed{\frac{a_0}{2}+}\sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$	$A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$ $(n=0,1,2,\cdots)$	$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$ $(n=0,1,2,\cdots)$ $b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$ $(n=1,2,\cdots)$

Some nonhomogeneous problems.

__Example__

$$\begin{cases} u_t = k u_{xx} & (0<x<L,\ t>0) \\ u(0,t)=A,\quad u(L,t)=B & (t>0) \\ u(x,0)=f(x) & (0<x<L) \end{cases}$$

Let $u_E(x) = A + \frac{B-A}{L}x.$ $\quad u_E(0)=A,\ u_E(L)=B$

$v(x,t) = u(x,t) - u_E(x).$ $\quad \boxed{u(x,t) = v(x,t) + u_E(x)}$

Then.

$$\begin{cases} v_t = k v_{xx} \\ v(0,t)=0,\quad v(L,t)=0 \\ v(x,0)=f(x)-u_E(x). \end{cases}$$

Example
$$\begin{cases} u_t = k u_{xx} + Q(x,t) & (0<x<L,\ t>0) \\ u(0,t) = A(t), \quad u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

Let $r(x,t) = A(t) + \frac{x}{L}\left[B(t) - A(t)\right]$

$v(x,t) = u(x,t) - r(x,t)$. $\boxed{u(x,t) = v(x,t) + r(x,t)}$

$$\begin{cases} \frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + \overline{Q}(x,t), \\ v(0,t) = 0,\ v(L,t) = 0 \\ v(x,0) = f(x) - r(x,0) \equiv g(x) \end{cases}$$

$\overline{Q}(x,t) = Q(x,t) - \frac{\partial r}{\partial t} + k\frac{\partial^2 r}{\partial x^2}$

$$\begin{cases} \phi_n'' + \lambda_n \phi_n = 0 & (0, L). \\ \phi_n(0) = \phi_n(L) = 0 \end{cases}$$

$\phi_n(x) = \sin\frac{n\pi x}{L}$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2$

$(n = 1, 2, 3, \cdots)$

Expansion

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)}$$

$g(x) = v(x,0) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x)$.

So, $\boxed{a_n = \int_0^L g(x)\phi_n(x)\,dx \Big/ \int_0^L \phi_n^2(x)\,dx. \quad n = 1, 2, \cdots}$

$v_t = \sum_1^{\infty} a_n'(t)\phi_n(x)$

$v_{xx} = \sum_1^{n} a_n(t)\phi_n'' = \sum_1 a_n(t)(-\lambda_n\phi_n)$

$v_t = kv_{xx} + \overline{Q}(x,t)$:

$$\sum_1^{\infty} a_n'(t)\phi_n(x) = \sum_1^{\infty} a_n(t)(-k\lambda_n)\phi_n(x) + \underbrace{\sum_{n=1}^{\infty} \overline{q}_n(t)\phi_n(x)}_{=\overline{Q}(x,t)}$$

$$\boxed{\overline{q}_n(t) = \int_0^L \overline{Q}(x,t)\phi_n(x)\,dx \Big/ \int_0^L \phi_n^2(x)\,dx, \quad n = 1, 2, \cdots}$$

$$\sum_{n=1}^{\infty}\left(a_n' + \kappa\lambda_n a_n - \bar{q}_n(t)\right)\phi_n(x) = 0.$$

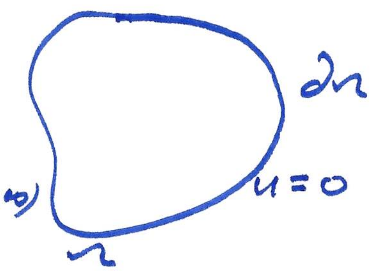
$\{\phi_n\}$ is complete. So,

$$\boxed{\begin{aligned} &a_n' + \kappa\lambda_n a_n = \bar{q}_n(t). \qquad n=1,2,\dots \\ &a_n(0) = \int_0^L g(x)\phi_n(x)\,dx \Big/ \int_0^L \phi_n^2(x)\,dx. \qquad n=1,2,\dots \end{aligned}}$$

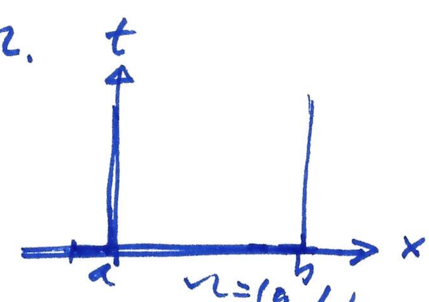
$$\boxed{\begin{aligned} a_n(t) = a_n(0)e^{-\lambda_n\kappa t} + e^{-\lambda_n\kappa t}\int_0^t \bar{q}_n(s)e^{\lambda_n\kappa s}\,ds \\ n=1,2,\dots \end{aligned}}$$

The method of separation of variables and the method of eigenvalues/eigenfunctions expansion for general domains

Let $\Omega \subset \mathbb{R}^d$ be bounded, smooth, and open, connected.



Consider
$$\begin{cases} u_t - \kappa\Delta u = 0 & \text{in } \cancel{t}\ \Omega\times(0,\infty) \\ u(x,0) = g(x) & \cancel{\text{on } \partial\Omega}\ \text{in } \Omega \\ u(x,t) = 0 & \text{on } \partial\Omega. \end{cases}$$



Separation of variables

$$u(x,t) = X(x)\,T(t)$$

$$X T' = \kappa \Delta X\, T$$

$$\frac{T'}{\kappa T} = \frac{\Delta X}{X} = -\lambda$$

$$\begin{cases} \Delta X + \lambda X = 0 & \text{in } \Omega \\ X = 0 & \text{on } \partial\Omega \end{cases}$$

$$T' = -\lambda\kappa T$$

$$\boxed{T(t) = Ce^{-\lambda\kappa t}}$$

<u>Theorem</u> (Spectrum for $-\Delta$ with the Dirichlet boundary condition)

Consider $\begin{cases} -\Delta\phi = \lambda\phi & \Omega, \\ \phi = 0 & \partial\Omega. \end{cases}$ $\left[\begin{matrix} \phi = \phi(x), & x \in \Omega \\ \lambda \in \mathbb{R} \end{matrix}\right]$.

This problem has infinitely many solution. (λ, ϕ).

(1) The eigenvalues λ are: $0 < \lambda_1 < \lambda_2 < \cdots$ $\left(\lim_{n\to\infty} \lambda_n = \infty.\right)$

(2) The first (or principal) eigenvalue λ_1 is a simple eigenvalue (or an eigen value with multiplicity 1).

Any eigenfunctions associated with λ_1 does not change sign in Ω.

(3) Each eigenvalue λ_n $(n \geq 1)$ has at most finitely many linearly indep. eigenfunctions. Eigenfunctions ~~(4)~~ corresponding to different eigenvalues are $L^2(\Omega)$ orthogonal. e.g.,

$$\left.\begin{matrix} \Delta\phi + \lambda\phi = 0 \\ \Delta\psi + \mu\psi = 0 \\ \lambda \neq \mu \end{matrix}\right. \implies \int_\Omega \phi\psi\, dx = 0.$$

(4) Eigenfunctions can form a complete ~~set of~~ orthognormal system in $L^2(\Omega)$.

i.e., we can choose 1 e-function for λ_1, n_1 (orthogonal) e-functions for λ_2 (n_1 is maximal), n_2 (orthogonal) e-funtions for λ_3, etc.

Use notation: $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$

$\phi_1, \phi_2, \phi_3, \ldots$

$$\langle \phi_i, \phi_j \rangle = \int_\Omega \phi_i \phi_j \, dx = \delta_{ij} \quad (i, j = 1, 2, \ldots).$$

Then for any $f \in L^2(\Omega)$

$$f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \qquad \langle f, \phi_n \rangle = \int_\Omega f \phi_n \, dx.$$

(5) For eigen pair (λ, ϕ), we have the Rayleigh quotient:

$$\lambda = \frac{\int_\Omega |\nabla \phi|^2 dx}{\int_\Omega \phi^2 dx}.$$

[This proves $\lambda > 0$.].

Back to
$$\begin{cases} u_t - k \Delta u = 0 & \Omega \times (0, \infty) \\ u = 0 & \partial\Omega \times (0, \infty) \\ u(x, 0) = g(x) & \Omega. \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) T_n(t) \qquad T_n(t) = e^{-\lambda_n k t}.$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n k t} \phi_n(x)}$$

$$g(x) = u(x, 0) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

$$\boxed{a_n = \langle g, \phi_n \rangle = \int_\Omega g(x) \phi_n(x) dx \quad (n = 1, 2, \ldots)}$$

For inhomogeneous equation with initial/boundary conditions:

$$\begin{cases} u_t = k \Delta u + f(x,t) & \Omega \times (0, \infty) \\ u = 0 & \partial\Omega \times (0, \infty) \\ u(x, 0) = g(x) & \Omega. \end{cases}$$

We use the method of eigenfunctions expansion.

Let $\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)}$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\,\phi_n(x)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} a_n'(t)\,\phi_n(x)$$

$$\Delta u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\Delta\phi_n(x) = \sum_{n=1}^{\infty} (-\lambda_n)\,a_n(t)\,\phi_n(x)$$

since $\Delta\phi_n + \lambda_n\phi_n = 0$.

$$u_t - k\Delta u = f$$

$$\Longrightarrow \sum_{n=1}^{\infty} \left(a_n'(t) + k\lambda_n a_n - f_n(t)\right)\phi_n(x) = 0$$

$$\boxed{a_n' + k\lambda_n a_n = f_n \qquad (n = 1, 2, \cdots)}$$

$$g(x) = u(x,0) = \sum_{n=1}^{\infty} a_n(0)\,\phi_n(x)$$

$$\boxed{a_n(0) = \langle g, \phi_n\rangle = \int_\Omega g(x)\,\phi_n(x)\,dx \qquad n = 1, 2, \cdots}$$

$$\boxed{a_n(t) = e^{-k\lambda_n t}\left[a_n(0) + \int_0^t e^{k\lambda_n s} f_n(s)\,ds\right] \qquad n = 1, 2, \cdots}$$