Section 1.2 Fundamental Solutions / Fourier Transforms

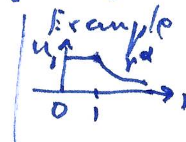
This part is for the pure initial-value problem in the entire space $\mathbb{R}^n$.

$$\begin{cases} u_t = D\Delta u + f(x,t), & (x \in \mathbb{R}^n,\ t>0) \\ u(x,0) = \phi(x) & (x \in \mathbb{R}^n) \end{cases}$$

Here $D>0$: const. $f(x,t)$: given. $\phi(x)$: given

We shall take $D=1$. [otherwise, change the time scale $t \longmapsto t' = Dt$.]

We first study Fourier transforms.



$u \in L^p(\mathbb{R}^n)$ p, α?

<u>Definition</u> Let $u \in L^1(\mathbb{R}^n)$ (i.e., $\int_{\mathbb{R}^n} |u(x)|\,dx < \infty$ and u is measurable) The Fourier transform (FT) of u is a function $\hat{u} = \hat{u}(\xi)$ $(\xi \in \mathbb{R}^n)$:

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx \quad (\xi \in \mathbb{R}^n).$$

If $v = v(\xi) \in L^1(\mathbb{R}^n)$ then the inverse FT of $v = v(\xi)$ is a function $\check{v} = \check{v}(x)$ $(x \in \mathbb{R}^n)$:

$$(\mathcal{F}^{-1}v)(x) = \check{v}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} v(\xi)\,d\xi \quad (x \in \mathbb{R}^n)$$

<u>Plancherel Theorem</u> If $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $\hat{u}, \check{u} \in L^2(\mathbb{R}^n)$ and

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}.$$

Here

$$\|u\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^2\,dx \right)^{1/2}.$$

Clearly. FT and inverse FT are linear

$$\mathcal{F}[\alpha u+\beta v] = \alpha\mathcal{F}[u] + \beta\mathcal{F}[v].$$

i.e. $\widehat{(\alpha u+\beta v)} = \alpha\hat{u} + \beta\hat{v}$.

Similarly. $\widecheck{\alpha u+\beta v} = \alpha\check{u} + \beta\check{v}$.

<u>Some other basic properties</u>

(1) $u = \check{\hat{u}}$. $\left(\cancel{\check{\hat{v}} = \hat{\check{v}}}.\right) \Rightarrow u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{u}(\xi)d\xi$.

$\hat{\check{u}} = u$

(2) $\langle u, v\rangle = \langle \hat{u}, \hat{v}\rangle$ $\quad (\Rightarrow \|u\|_{L^2} = \|\hat{u}\|_{L^2})$.

Here $\langle u, v\rangle = \int_{\mathbb{R}^n} u(x)\overline{v(x)}\,dx$

$\overline{v(x)}$ means the complex conjugate of $v(x)$.

(3) $\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi) \quad \forall \xi \in \mathbb{R}^n$.

Here $\alpha = (\alpha_1, \cdots, \alpha_n)$, each $\alpha_j \geq 0$ is an integer.

$$D^\alpha u(\xi) = \frac{\partial^{\alpha_1+\cdots+\alpha_n} u(\xi)}{\partial^{\alpha_1}\xi_1\,\partial^{\alpha_2}\xi_2\cdots\partial^{\alpha_n}\xi_n}. \qquad \xi = (\xi_1, \xi_2, \cdots, \xi_n)$$

Notation $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

(4) $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$:

$$\widehat{u*v} = (2\pi)^{n/2}\hat{u}\,\hat{v}$$

convolution:

$$(u*v)(x) = \int_{\mathbb{R}^n} u(x-y)\,v(y)\,dy \quad \forall x \in \mathbb{R}^n$$

<u>Example</u> $\quad u = u(x_1, x_2)$. $\quad u = 0$ at ∞. $\quad \partial_{x_1} u = 0$ at ∞.

$$\widehat{\partial_{x_1} u(x_1, x_2)} = \widehat{\partial_1 u}(\xi_1, \xi_2)$$

$$= \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{-i(\xi_1 x_1 + \xi_2 x_2)}\partial_{x_1} u(x_1, x_2)\,dx_1\,dx_2$$

$$= -\frac{1}{2\pi}\iint_{\mathbb{R}^2} \partial_{x_1}\left(e^{-i(\xi_1 x_1+\xi_2 x_2)}\right) u(x_1, x_2)\,dx_1\,dx_2$$

$$= -\frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{-ix\cdot\xi}(-i\xi_1)\,u(x)\,dx$$

$$= i\xi_1 \;\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi} u(x)\,dx$$

$$= i\xi_1\,\hat{u}(\xi).$$

<u>Example</u>

$$\widehat{u*v}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} (u*v)(x)\,e^{-ix\cdot\xi}\,dx$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\left[\int_{\mathbb{R}^n} u(x-y)\,v(y)\,dy\right]dx$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} e^{-i(x-y)\cdot\xi}u(x-y)\,e^{-iy\cdot\xi}v(y)\,dy\right]dx$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}v(y)\underbrace{\left[\int_{\mathbb{R}^n} e^{-i(x-y)\cdot\xi}u(x-y)\,dx\right]}_{=\int_{\mathbb{R}^n} e^{-iz\cdot\xi}u(z)\,dz.}dy$$

$$= \int_{\mathbb{R}^n} e^{-iy\cdot\xi}v(y)\,\hat{u}(\xi)\,dy$$

$$= \hat{u}(\xi)\cdot(2\pi)^{n/2}\;\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}v(y)\,dy$$

$$= (2\pi)^{n/2}\,\hat{u}(\xi)\cdot\hat{v}(\xi).$$

Use this ($\widehat{u*v} = \hat{u}\hat{v}$) to show $\langle u,v\rangle = \langle\hat{u},\hat{v}\rangle$.

Use the Fourier transform to solve

$$\begin{cases} u_t - \Delta u = 0 & (x \in \mathbb{R}^n, t > 0) \\ u(x,0) = f(x) & (x \in \mathbb{R}^n) \end{cases}$$

FT: $\begin{cases} \hat{u}_t + |\xi|^2 \hat{u} = 0 & (t > 0, \ \xi \in \mathbb{R}^n) \\ \hat{u} = \hat{f} & t = 0. \end{cases}$

$$\widehat{\Delta u}(\xi) = \sum_{k=1}^n \widehat{\partial^2_{x_k} u}(\xi) = \sum_{k=1}^n (i\xi_k)^2 \hat{u}(\xi)$$

$$= -\left(\sum_{k=1}^n \xi_k^2\right) \hat{u}(\xi) = -|\xi|^2 \hat{u}(\xi).$$

ODE (initial-value problem) $\begin{cases} \hat{u}_t + |\xi|^2 \hat{u} = 0 \\ \hat{u}(t=0) = \hat{f} \end{cases}$

$$\hat{u}(\xi, t) = \hat{f}(\xi) e^{-t|\xi|^2} \quad (\forall \xi \in \mathbb{R}^n, \ t \ge 0)$$

So, $u(x,t) = \check{\hat{u}}(\xi,t) = \check{\overbrace{\hat{f}(\xi) e^{-t|\xi|^2}}} = \mathcal{F}^{-1}\left(\hat{f}(\xi) e^{-t|\xi|^2}\right)$

Hence $u(x,t) = \dfrac{f(x) * \mathcal{F}^{-1}\left(e^{-t|\xi|^2}\right)}{(2\pi)^{n/2}}$

[Let $F(x,t) = \mathcal{F}^{-1}(e^{-t|\xi|^2})$. Then

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F(x-y,t) f(y)\, dy$$]

Or, let $K(x,t) = \dfrac{1}{(2\pi)^{n/2}} \mathcal{F}^{-1}\left(e^{-t|\xi|^2}\right)(x)$ then

$$\boxed{u(x,t) = \int_{\mathbb{R}^n} K(x-y,t) f(y)\, dy}$$

We need to compute $K(x,t)$.

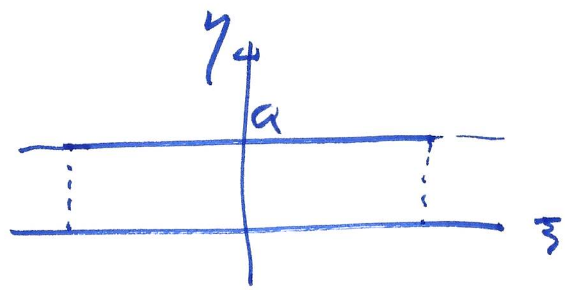
$$K(x,t)=\int_{\mathbb{R}^n} e^{ix\xi}e^{-t|\xi|^2}\,d\xi \Big/ (2\pi)^n$$

$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{n} e^{ix_k\xi_k - t\xi_k^2}\,d\xi_1\cdots d\xi_n \Big/ (2\pi)^n$$

$$=\prod_{k=1}^{n}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix_k\xi_k - t\xi_k^2}\,d\xi_k$$

So, only one-dimension: $n=1$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi - t\xi^2}\,d\xi.$$



$\xi \to \xi + ia$ $$=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix(\xi+ia)-t(\xi+ia)^2}\,d\xi$$

$a\in\mathbb{R},\ a\neq 0.$

Find a: $ix(\xi+ia)-t(\xi+ia)^2$

$$=ix\xi - xa - t\xi^2 - 2ti\xi a + ta^2$$

$$=i\xi\underbrace{(x-2ta)} - xa - t\xi^2 + ta^2$$

So, $a=\frac{x}{2t}$. (So, only for $t>0$)

the exponent: $ix(\xi+ia-t(\xi+ia)^2$

$$=-xa-t\xi^2+ta^2$$

$$=-\frac{x^2}{2t}-t\xi^2+\frac{tx^2}{4t^2}=-t\xi^2-\frac{x^2}{4t}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi-t\xi^2}\,d\xi=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t\xi^2-\frac{x^2}{4t}}\,d\xi$$

$$= \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} e^{-t\eta^2} d\eta$$

$$\underline{t\eta^2 = \xi^2} \quad \frac{1}{2\pi} e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} e^{-\xi^2} t^{-\frac{1}{2}} d\xi$$

$\xi = \sqrt{t}\,\eta$

$\eta = \xi t^{-\frac{1}{2}}$

$d\eta = t^{-\frac{1}{2}} d\xi$

$$= \frac{1}{\sqrt{4\pi^2 t}} e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$. since: $I = \int_{-\infty}^{\infty} e^{-\xi^2} d\xi$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dx\,dy$$

$$= 2\pi \int_0^{\infty} e^{-r^2} r\,dr = \pi \left(-e^{-r^2}\right)\Big|_0^{\infty} = \pi.$$

Now, $\boxed{K(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \quad (t>0,\ x\in\mathbb{R}^n)}$

$$\begin{cases} u_t - \Delta u = 0 & \mathbb{R}^n \times (0,\infty) \\ u = f & \mathbb{R}^n \times \{0\}. \end{cases}$$

$\Rightarrow$ $\boxed{u(x,t) = \int_{\mathbb{R}^n} K(x-y) f(y) dy.}$

Call $K(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$ the fundamental solution of ~~the~~ heat eq: $u_t = \Delta u$.
[Or the Gaussian Kernel]

Some properties of $K(x)$

(1) $K(x,t) \in C^\infty$ for $x \in \mathbb{R}^n$ and $t > 0$. [Obvious]
$K(-x,t) = K(x,t). = g(|x|, t)$.

(2) $K_t - \Delta K = 0$ for $t > 0$ [Direct verification]

(3) $K > 0$ for $t > 0$. [Obvious]

(4) $\int_{\mathbb{R}^n} K(x,t)\,dx = 1 \quad \forall t > 0$

Use Thm below with $f(x) \equiv 1 \Rightarrow u \equiv 1$.
$1 = u(0,t)$
$= \int K(-y,t)\,dy$
$= \int K(x,t)\,dx$

(5) For any $\varepsilon > 0$
$$\lim_{t \to 0^+} \int_{|x|>\varepsilon} K(x,t)\,dx = 0$$

~~{uniform convergence}~~

$$\left[\int_{|x|>\varepsilon} K(x,t)\,dx = C \int_{|y|>\varepsilon/\sqrt{t}} e^{-\frac{|y|^2}{4}}\,dy \to 0 \right]$$

Theorem If $f = f(x)$ is continuous and bounded in $\mathbb{R}^n$ then

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t) f(y)\,dy$$
$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy$$

is a C^∞-function for $x \in \mathbb{R}^n$ and $t > 0$, satisfying

~~and~~ $$\begin{cases} u_t - \Delta u = 0 & (x \in \mathbb{R}^n,\ t > 0). \\ u(x,0) = f(x) & (x \in \mathbb{R}^n). \end{cases}$$

Interpretation of the fundamental solution:

$$\begin{cases} K_t - \Delta K = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ K = \delta_0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Or. Extend K to be 0 for $t<0$. Then

$$K_t - \Delta K = \delta \quad \text{in } \mathbb{R}^{n+1}$$

i.e.,

$$\int_{\mathbb{R}^{n+1}} K(x,t)\,(-\partial_t - \Delta)\phi \, dx\,dt = \phi(0,0) \quad \forall \phi \in C_c^\infty(\mathbb{R}^{n+1}).$$

Proof. Let $\varepsilon > 0$ and denote

$$K_\varepsilon(x,t) = \begin{cases} K(x,t) & \text{if } t > \varepsilon, \\ 0 & \text{if } t \le \varepsilon. \end{cases}$$

Then $K_\varepsilon \to K$ in the sense

$$\int_{\mathbb{R}^{n+1}} K_\varepsilon \eta \, dx\,dt \to \int_{\mathbb{R}^{n+1}} K \eta \, dx\,dt \quad \forall \eta \in C_c^\infty(\mathbb{R}^{n+1}).$$

Now

$$\int K_\varepsilon(-\partial_t - \Delta)\phi\,dx\,dt$$

$$= \int_\varepsilon^\infty \left(\int_{\mathbb{R}^n} K(x,t)(-\partial_t - \Delta)\phi(x,t)\,dx \right) dt$$

$$\overset{\text{Integration by parts}}{=} \int_\varepsilon^\infty \int_{\mathbb{R}^n} (\partial_t - \Delta) K(x,t) \phi(x,t)\,dx \Big) dt$$

$$\cancel{= \int_\varepsilon^\infty \int_{\mathbb{R}^n} \partial_t - \Delta} + \int_{\mathbb{R}^n} K(x,\varepsilon)\phi(x,\varepsilon)\,dx$$

But $(\partial_t - \Delta)K(x,t) = 0$ for $t > \varepsilon$.

$K(x,\varepsilon) \to \delta_0(x)$ as $\varepsilon \to 0$. Q.E.D.

Before we move on to solving inhomogeneous heat equations $u_t - \Delta u = f(x,t)$, we discuss some basic observations of solutions to $u_t = \Delta u$ and rederive the Gaussian kernel
$$K(x,t) = \frac{1}{(\sqrt{4\pi t})^n} e^{-\frac{|x|^2}{4t}} \quad (x \in \mathbb{R}^n, t > 0).$$

Consider
$$u_t = \Delta u \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

(1) Linearity: u, v are solutions, $\alpha, \beta \in \mathbb{R}$
$\Longrightarrow \alpha u + \beta v$ are solutions

(2) If u solves the equation, then any partial derivative (of any order) is also a solution.

(4) Translational invariance: $u = u(x,t)$ solves the equation, $t_0 > 0$, $x_0 \in \mathbb{R}^n$, $v(x,t) = u(x - x_0, t - t_0)$ solves $v_t = \Delta v$ $(x \in \mathbb{R}^n, t > t_0)$.

(3) If $S(x,t)$ solves $S_t = \Delta S$ then
$$v(x,t) = \int_{\mathbb{R}^n} S(x-y,t)\, g(y)\, dy$$
also solves $v_t = \Delta v$ for ANY $g = g(y)$.

(5) Important scaling: If $u = u(x,t)$ solves $u_t = \Delta u$, $\lambda > 0$, then $v(x,t) = u(\sqrt{\lambda} x, \lambda t)$ also solves $v_t = \Delta v$.

Rederive the Gaussian kernel.

Let $u(x,t) = t^\alpha g\left(\frac{|x|}{\sqrt{t}}\right)$. $|x| = \sqrt{\sum_{k=1}^n x_k^2}$, α = const. to be determined.

Assume $u_t = \Delta u$.

$$u_t = \alpha t^{\alpha-1} g + t^\alpha g' \cdot |x| \left(-\frac{1}{2} t^{-3/2}\right)$$
$$= \alpha t^{\alpha-1} g - \frac{1}{2} t^{\alpha-3/2} |x|\, g'$$

$$u_{x_j} = t^\alpha g' \cdot \frac{1}{\sqrt{t}} \frac{x_j}{|x|} = t^{\alpha-1/2} \frac{x_j}{|x|} g'$$

$$u_{x_j x_j} = t^{\alpha-1/2} \frac{|x| - x_j^2/|x|}{|x|^2} g'$$
$$+ t^{\alpha-1/2} \frac{x_j}{|x|} g'' \cdot \frac{1}{\sqrt{t}} \frac{x_j}{|x|}$$
$$= t^{\alpha-1/2} \left(\frac{|x|^2 - x_j^2}{|x|^3}\right) g' + t^{\alpha-1} \frac{x_j^2}{|x|^2} g''$$

$u_t = \Delta u$

$$\Longrightarrow \quad \alpha t^{\alpha-1} g - \frac{1}{2} t^{\alpha-3/2} |x|\, g'$$
$$= t^{\alpha-1/2} \frac{(n-1)}{|x|} g' + t^{\alpha-1} g''$$

Let $p = |x|/\sqrt{t}$.

$$(*) \qquad \underbrace{g'' + \frac{n-1}{p} g'}_{\text{These look familiar.}} + \frac{1}{2} p g' - \alpha g = 0.$$

$$g'' + \frac{n-1}{p} g' = p^{1-n} \left(p^{n-1} g'\right)'.$$

So, Multiply $(*)$ by p^{n-1}

$$\left(p^{n-1} g'' + (n-1) p^{n-2} g'\right) + \left(\frac{1}{2} p^n g' - \alpha p^{n-1} g\right) = 0$$
$$\left(p^{n-1} g'\right)' + \left(\frac{1}{2} p^n g' - \alpha p^{n-1} g\right) = 0$$

choose α to make the second term a derivative.

Try $\frac{1}{2}\rho^n g' - \alpha \rho^{n-1} g = \frac{1}{2}(\rho^n g)'$.

so, $\frac{1}{2}\rho^n g' - \alpha\rho^{n-1} g = \frac{1}{2}\rho^n g' + \frac{n}{2}\rho^{n-1} g$

$$-\alpha = n/2. \qquad \boxed{\alpha = -n/2}$$

Now, (*) is

$$(\rho^{n-1} g')' + \frac{1}{2}(\rho^n g)' = 0$$

$$\rho^{n-1} g' + \frac{1}{2}\rho^n g = A_1 = \text{const.}$$

Let just $A_1 = 0$.

$$g' + \frac{1}{2}\rho g = 0 \qquad \frac{dg}{d\rho} = -\frac{1}{2}\rho g$$

$$\Rightarrow \quad g = g(\rho) = C e^{-\rho^2/4}.$$

$$\boxed{u(x,t) = C t^{-n/2} e^{-\frac{|x|^2}{4t}}}.$$

We finish this section by deriving a formula for solutions of $u_t = \Delta u + f(x,t)$.

Decompose $\begin{cases} u_t = \Delta u + f(x,t) & (x \in \mathbb{R}^n, t > 0) \\ u(x,0) = \phi(x) & (x \in \mathbb{R}^n) \end{cases}$

into $\begin{cases} v_t = \Delta v + f(x,t) \\ v(x,0) = 0 \end{cases}$ and $\begin{cases} w_t = \Delta w \\ w(x,0) = \phi(x). \end{cases}$

$u(x,t) = v(x,t) + w(x,t)$.

So, we may just assume $\phi(x) \equiv 0$ and study $\begin{cases} u_t = \Delta u + f(x,t) & (x \in \mathbb{R}^n, t > 0) \\ u(x,0) = 0 & (x \in \mathbb{R}^n) \end{cases}$

Duhamel's principle for $\begin{cases} u_t = \Delta u + f(x,t) & (x \in \mathbb{R}^n, t>0) \\ u(x,0)=0 & (x \in \mathbb{R}^n) \end{cases}$

Define for any $0 < s < t$.

$$u(x,t;s) = \int_{\mathbb{R}^n} K(x-y,t-s) f(y,s)\,dy$$

Clearly. $u(x,t;s)$ solves

$$\begin{cases} u_t(\cdot,\cdot;s) - \Delta u(\cdot,\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot,\cdot;s) = f(\cdot,s) & \text{in } \mathbb{R}^n \times \{t=s\} \end{cases}$$

at $t=s$.

Then

$$\boxed{u(x,t) = \int_0^t u(x,t;s)\,ds \quad (x \in \mathbb{R}^n,\ t \geq 0)}$$

i.e.,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) f(y,s)\,dy\,ds$$

$$= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy \right) ds$$

Finally. $\begin{cases} u_t = \Delta u + f(x,t) & (x \in \mathbb{R}^n, t>0) \\ u(x,0) = \phi(x) & (x \in \mathbb{R}^n) \end{cases}$

has the solution

$$\boxed{\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} K(x-y,t)\phi(y)\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) f(y,s)\,dy\,ds \end{aligned}}$$

From solutions to linear and homogeneous equations to solutions to linear and nonhomogeneous equations: More discussions

ODE. $\begin{cases} \frac{du}{dt} + Au = 0 \\ u(0) = u_0 \end{cases} \Rightarrow u(t) = S(t)u_0 = e^{-At}u_0$

$S(t) = e^{-At}.$

$$\begin{cases} \frac{du}{dt} + Au = f(t) \\ u(0) = u_0. \end{cases}$$

$u = u_1 + u_2.$ $\begin{cases} \frac{du_1}{dt} + Au_1 = f(t) \\ u_1(0) = 0 \end{cases}$ $\begin{cases} \frac{du_2}{dt} + Au_2 = 0 \\ u_2(0) = u_0. \end{cases}$

$u_2(t) = S(t)u_0.$

So, may assume $u_0 = 0$.

$$\begin{cases} \frac{du}{dt} + Au = f(t) \\ u(0) = 0 \end{cases}$$

The solution is $u(t) = e^{-At}\int_0^t f(s)e^{As}\,ds$

or $u(t) = \int_0^t S(t-s)f(s)\,ds$

How to obtain this solution? By the method of integrating factor.

$$u' + Au = f \Longrightarrow e^{At}(u' + Au) = e^{At}f(t)$$

$$\frac{d}{dt}\left(e^{At}u(t)\right)' = e^{At}f(t)$$

$$e^{As}u(s)\Big|_{s=0}^{s=t} = \int_0^t e^{As}f(s)\,ds$$

$$e^{At}u(t) - e^0\underbrace{u(0)}_{=0} = \int_0^t e^{As}f(s)\,ds$$

$$u(t) = \int_0^t e^{-A(t-s)}f(s)\,ds.$$

$$\boxed{u(t) = \int_0^t S(t-s)f(s)\,ds}$$

System of ODE.

$$\begin{cases} \dfrac{d\vec{u}}{dt} + A\vec{u} = 0 \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

$$\vec{u}(t) = e^{-At}\vec{u}_0$$

Let $S(t) = e^{-At}$

For each $t \geq 0$, $S(t)$ is a matrix.

$$\vec{u}(t) = S(t)\vec{u}_0$$

$$\vec{u} = \vec{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}, \quad \vec{u}_0 = \begin{bmatrix} u_{10} \\ u_{20} \\ \vdots \\ u_{n0} \end{bmatrix}$$

A: $n \times n$ constant matrix

$$\vec{f} = \vec{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

Now, nonhomogeneous: (but, with $\vec{u}_0 = \vec{0}$)

$$\begin{cases} \dfrac{d\vec{u}}{dt} + A\vec{u} = \vec{f}(t) \\ \vec{u}(0) = \vec{0} \end{cases}$$

Similar, $\cancel{\vec{u}(t) =}$ $e^{At}\left(\dfrac{d\vec{u}}{dt} + A\vec{u}\right) = e^{At}\vec{f}(t)$

$$\frac{d}{dt}\left(e^{At}\vec{u}(t)\right) = e^{At}\vec{f}(t)$$

$\int_0^t \cdots ds$: $\quad e^{At}\vec{u}(t) = \int_0^t e^{As}\vec{f}(s)\,ds$

Let $S(t) = e^{-At}$. So $S(-t) = e^{At}$.

Note $S(t)S(-t) = S(t-t) = S(0) = I$ (identity matrix)

So, $\quad \vec{u}(t) = \int_0^t e^{-A(t-s)}\vec{f}(s)\,ds$

$$\boxed{\vec{u}(t) = \int_0^t S(t-s)\vec{f}(s)\,ds}$$

Recovery of Duhamel's principle:

Let $\vec{u}(t;s) = S(t-s)\vec{f}(s)$ $\quad (0 \leq s \leq t)$. Then.

$$\begin{cases} \dfrac{d}{dt}\vec{u}(t;s) + A\vec{u}(t;s) = 0 & (t \geq s) \\ \vec{u}(s,s) = f(s) & \end{cases}$$

~~$0 \le s \le t$~~

Verify:

$$\frac{d}{dt}\vec{u}(t;s) = \frac{d}{dt}\left(S(t-s)\vec{F}(s)\right)$$
$$= \frac{d}{dt} e^{-At} \cdot e^{As}\vec{F}(s)$$
$$= -A e^{-At} e^{As}\vec{F}(s)$$
$$= -A S(t-s)\vec{F}(s) = -A\vec{u}(t;s)$$
$$\vec{u}(s;s) = S(0)\vec{F}(s) = \vec{F}(s).$$

So, the Duhamel's principle follows from the method of integrating factor. — More mathematical than physical/chemical?