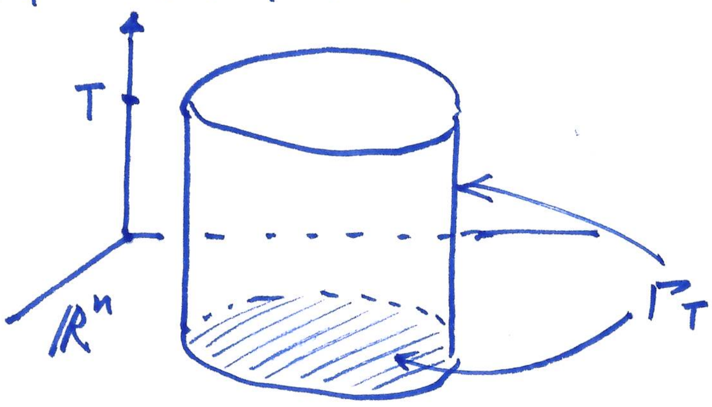
Section 2.2 Mean-Value Theorem. Maximum Principle. Energy Method

Notation: $\Omega \subseteq \mathbb{R}^n$: open, bounded

$T > 0$.

$\Omega_T = \{(x,t) : x \in \Omega,\ t \in (0,T]\} = \Omega \times (0,T]$

$\Gamma_T = \Omega \times \{0\} \cup \partial\Omega \times [0,T] = \overline{\Omega_T} \setminus \Omega_T$



Mean values for harmonic functions are averages over spheres: level sets of fundamental sol'n for Laplace's equation $\left(\frac{1}{4\pi|x-y|}\right)$.

Recall: the fundamental solution for $u_t - \Delta u = 0$ is

$$K(x,t) = \frac{1}{\sqrt{4\pi t}^{\,n}} e^{-\frac{|x|^2}{4t}}.$$

So, try to average of solution over level sets of $K(x,t)$.

For $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, denote

$$E(x,t;r) = \left\{ (y,s) : y \in \mathbb{R}^n,\ s \in \mathbb{R},\ s \le t,\ K(x-y,t-s) \ge \frac{1}{r^n} \right\}$$

Call it a "heat ball".

Notation

$C_1^2(\Omega_T) = \{ u = u(x,t) : \Omega_T \to \mathbb{R},\ u, u_t, \nabla u, \nabla^2 u \in C(\Omega_T)\}$

$u \in C_1^2(\Omega_T)$ means $u = u(x,t)$ is continuous in Ω_T and all first-order space and time partial derivatives and all second-order space partial derivatives.

Theorem (Mean Values) Let $u \in C_1^2(\Omega_T)$ solve the heat equation $u_t - \Delta u = 0$ in Ω_T. Then, for any $E(x,t;r) \subset \Omega_T$, we have

$$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\, dy\, ds.$$

Sketch of Proof Without loss of generality, set $(x,t) = (0,0)$. Denote $E(r) = E(0,0,r)$, and

$$\phi(r) = \frac{1}{r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dy\, ds$$

$$\overset{\substack{y \to ry \\ s \to r^2 s}}{=\!=} \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2}\, dy\, ds$$

Idea: ① show $\phi'(r) \equiv 0$.

② $\phi(r) = \lim_{t \to 0} \phi(t) = 4u(0,0)$

$$\phi'(r) = \iint_{E(1)} \left(\sum_{i=1}^n u_{y_i} y_i \frac{|y|^2}{s^2} + 2r u_s \frac{|y|^2}{s} \right) dy\, ds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left(\sum_{i=1}^n u_{y_i} y_i \frac{|y|^2}{s^2} + 2u_s \frac{|y|^2}{s} \right) dy\, ds$$

$$\equiv A + B$$

Introduce $\psi = -\frac{n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log r$.

$\psi = 0$ on $\partial E(r)$. So,

$$B = \frac{1}{r^{n+1}} \iint_{E(r)} 4 u_s \sum_{i=1}^{n} y_i \psi_{y_i}\, dy\, ds$$

Integration by parts (IBP)
$$= -\frac{1}{r^{n+1}} \iint_{E(r)} \left(4 n u_s \psi + 4 \sum_{i=1}^{n} u_{s y_i} y_i \psi \right) dy\, ds$$

IBP w.r.t. s
$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left(-4 n u_s \psi + 4 \sum_{i=1}^{n} u_{y_i} y_i \psi_s \right) dy\, ds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left(-4 n u_s \psi + 4 \sum_{i=1}^{n} u_{y_i} y_i \left(-\frac{n}{2s} - \frac{|y|^2}{4s^2} \right) \right) dy\, ds$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} \left(-4 n u_s \psi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i \right) dy\, ds - A.$$

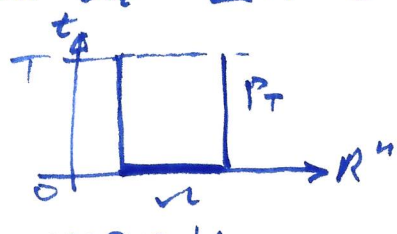
$$\phi'(r) = A + B \xrightarrow{u_s - \Delta u = 0} = \frac{1}{r^{n+1}} \iint_{E(r)} \left(-4n \Delta u\, \psi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i \right) dy\, ds$$

$$\text{IBP} \longrightarrow = \sum_{i=1}^{n} \iint_{E(r)} \left(4 n u_{y_i} \psi_{y_i} - \frac{2n}{s} u_{y_i} y_i \right) dy\, ds$$

$$= 0 \qquad \text{since } \psi_{y_i} = \frac{y_i}{2s}.$$

Now,
$$\phi(r) = \lim_{t \to 0} \phi(t) \overset{= u(0,0)}{=} \left(\lim_{t \to 0} \frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s}\, dy\, ds \right)$$
$$= u(0,0)\, 4. \qquad \underline{\text{Q.E.D.}}$$

<u>Theorem (~~Strong~~ Maximum Principle)</u> Assume $u \in C(\bar{\Omega}_T) \cap C_1^2(\Omega_T)$ solves the heat equation $u_t - \Delta u = 0$ in Ω_T. Then
$$\max_{\bar{\Omega}_T} u = \max_{\Gamma_T} u$$



Moreover, if Ω is connected and $\exists\, (x_0, t_0) \in \Omega_T$ s.t. $u(x_0, t_0) = \max_{\bar{\Omega}_T} u$ then $u = \text{const}$ in Ω_{t_0}.

<u>Sketch of Proof</u> Similar to that for Laplace's equation, use the mean-value Theorem.

If $u(x_0,t_0) = M \equiv \max_{\Omega_T} u$, then for $r>0$ small

$$M = u(x_0,t_0) = \frac{1}{4r^n} \iint_{E(x_0,t_0,r)} u(y,s) \frac{|x_0-y|^2}{(t_0-s)^2}\, dy\, ds \le M$$

since $1 = \frac{1}{4r^n} \iint_{E(x_0,t_0,r)} \frac{|x_0-y|^2}{(t_0-s)^2}\, dy\, ds$.

So, $u \equiv M$ on $E(x_0,t_0,r)$.

If $(y_0,s_0) \in \Omega_T$, $s_0 < t_0$ and $\exists$ line segment L connecting (y_0,s_0) and (x_0,t_0), then $u \equiv M$ on L. This is because, if we set

$$r_0 := \min\{s \ge s_0 : u(x,t) = M \text{ for all } (x,t) \in L,\ s \le t \le t_0\},$$

then $r_0 = s_0$.

The last part: use the connectedness of Ω. Connect (x_0,t_0) to any (y_0,s_0), with $s_0 < t_0$, by finitely many line segments. <u>Q.E.D.</u>

<u>Thm</u> (<u>Weak Maximum Principle</u>) Let $u \in C(\overline{\Omega_T}) \cap C_1^2(\Omega_T)$ be such that $\Delta u \ge u_t$ in Ω_T ($\Delta u \le u_t$ in Ω_T). Then u achieves its maximum on Γ_T (u achieves its minimum on Γ_T).

Basic idea of Proof

Assume for a moment $\Delta u > u_t$ in Ω_T. If u reaches its maximum at some interior point of Ω_T: (x,t), $x \in \Omega$, $0 < t < T$, then $\partial_t u(x,t) = 0$, $\nabla u(x,t) = 0$ and $\Delta u(x,t) \le 0$ (since each $u_{x_i x_i}(x,t) \le 0$), Contradicting $\Delta u > u_t$.

Hence $\max\limits_{\Omega_\tau} u = \max\limits_{\Gamma_\tau} u$ for any $0 < \tau < T$.

Let $\tau \to T$. Then, we can replace τ by T.

Now, just assume $\Delta u \ge u_t$ in Ω_T. Introduce $v = u - kt$ for $k > 0$. Thus, $v \le u$ in $\bar{\Omega}_T$, and $\Delta v - v_t = \Delta u - u_t + k > 0$. Hence, by what is proved above,

$$\max_{\bar{\Omega}_T} u = \max_{\bar{\Omega}_T} (v + kt) \le \max_{\bar{\Omega}_T} v + kT$$

$$= \max_{\Gamma_T} v + kT = \max_{\Gamma_T} u + kT.$$

Let $k \to 0$. Q.E.D.

Theorem (Uniqueness) There exists at most one solution to

$$\begin{cases} u_t - \Delta u = f(x,t) & \Omega \times (0,T] \\ u(x,t) = g(x,t), & x \in \partial\Omega, \ t > 0. \\ u(x,0) = \phi(x), & x \in \Omega. \end{cases}$$

Pf. If v, w both solve the problem, then $u = v - w$ solves

$$\begin{cases} u_t - \Delta u = 0 & \Omega \times (0,T] \\ u(x,t) = 0 & x \in \partial\Omega, \ t > 0 \\ u(x,0) = 0 & x \in \Omega \end{cases}$$

So, $u = 0$ on Γ_T.

Max. Principle $\Longrightarrow$ $u \le 0$ in Ω_T.

Apply MP to $-u$. $((-u)_t - \Delta(-u) = 0$, $-u = 0$ on $\Gamma_T)$

$\Longrightarrow -u \le 0$. So, $u \equiv 0$ Q.E.D.

<u>Theorem</u> (Maximum principle for the Cauchy problem) Suppose $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

and satisfies the growth estimate

$$u(x,t) \le A e^{a|x|^2} \quad (x \in \mathbb{R}^n,\ 0 \le t \le T)$$

for some constants $A, a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g .$$

Note: Without the growth assumption, this can fail.

<u>Theorem</u> (Uniqueness for the Cauchy problem) Let $f \in C(\mathbb{R}^n \times [0,T])$ and $g \in C(\mathbb{R}^n)$. Then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ of ~~the~~ the initial-value problem

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

satisfying the growth estimate

$$|u(x,t)| \le A e^{a|x|^2} \quad (x \in \mathbb{R}^n,\ 0 \le t \le T)$$

for some consts $A > 0$ and $a > 0$.

Note: without the growth assumption, the sol'n is not unique!

For the initial-boundary-value problem of the (nonhomogeneous) heat equation, we can prove the solution by the energy method.

Basically, if $\begin{cases} u_t - \Delta u = 0 & \Omega \times (0,T] \\ u(x,t) = 0 & x \in \partial\Omega,\ t \in (0,T] \\ u(x,0) = 0 & x \in \Omega. \end{cases}$

then we want to show $u \equiv 0$ in Ω.
Here we assume $u \in C(\bar{\Omega}_T) \cap C_1^2(\Omega_T)$.

Set $\quad e(t) = \int_\Omega u(x,t)^2 dx \qquad 0 \le t \le T.$

Then

$$\begin{aligned} e'(t) &= \int_\Omega 2u\,u_t\,dx \\ &= 2\int_\Omega u\,\Delta u\,dx \\ &= -2\int_\Omega |\nabla u|^2 dx + 2\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,dS. \\ \text{since } u=0 \text{ on } \partial\Omega \quad &= -2\int_\Omega |\nabla u|^2 dx \le 0 \end{aligned}$$

So, $e(t) \le e(0) = 0$. So, $e(t) = 0 \ \forall t \in [0,T]$.
Hence, $u(x,t) \equiv 0 \ \forall x \in \Omega,\ \forall t \in [0,T]$.

<u>Theorem</u> (Backward Uniqueness) Let v, w be in $C_1^2(\Omega_T) \cap C(\bar{\Omega}_T)$. Suppose

$\begin{cases} v_t - \Delta v = f(x,t), & \Omega_T \\ v = g & \text{on } \partial\Omega \times [0,T] \end{cases} \qquad \begin{cases} w_t - \Delta w = f(x,t), & \Omega_T \\ w = g & \text{on } \partial\Omega \times [0,T] \end{cases}$

Suppose $v(x,T) = w(x,T)$ $(x \in \Omega)$. Then $v \equiv w$ on Ω_T.

Proof Set $u = v - w$. Then $\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u(x,T) = 0 & \forall x \in \Omega. \end{cases}$

Let
$$e(t) = \int_\Omega u(x,t)^2 dx \quad (0 \le t \le T)$$

Then $\dot{e}(t) \underset{\uparrow}{=} -2 \int_\Omega |\nabla u|^2 dx$
as before

$$\ddot{e}(t) = -4 \int_\Omega \nabla u \cdot \nabla u_t \, dx = 4 \int_\Omega \Delta u \, u_t \, dx$$
$$= 4 \int_\Omega (\Delta u)^2 dx$$

Since $u = 0$ on $\partial\Omega$,
$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega u \, \Delta u \, dx \le \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega (\Delta u)^2 dx\right)^{\frac{1}{2}}$$

So,
$$\left(\dot{e}(t)\right)^2 = 4 \left(\int_\Omega |\nabla u|^2 dx\right)^2$$
$$\le \int_\Omega u^2 dx \cdot 4 \int_\Omega (\Delta u)^2 dx$$
$$= e(t)\, \ddot{e}(t)$$
i.e.,
$$\ddot{e}(t) e(t) \ge \left(\dot{e}(t)\right)^2.$$

If $e(t) = 0$ for all $t \in [0,T]$, then we are done.
Otherwise, $\exists\, [t_1, t_2] \subseteq [0,T]$ with
$$e(t) > 0, \quad t_1 \le t < t_2, \quad e(t_2) = 0$$
With these, we define $f(t) \equiv \log e(t)$, $t \in [t_1, t_2)$.
and get
$$\ddot{f}(t) = \left(\frac{\dot{e}(t)}{e(t)}\right)^{\cdot} = \frac{\ddot{e} e - \dot{e}^2}{e(t)^2} \ge 0$$
So, f is convex on (t_1, t_2). Therefore, if $0 < \tau < 1$, $t_1 < t < t_2$,
then

$$f((1-\tau)t_1 + \tau t) \le (1-\tau) f(t_1) + \tau f(t).$$

$$\log e((1-\tau)t_1 + \tau t) \le (1-\tau) \log e(t_1) + \tau \log e(t)$$

$$e((1-\tau)t_1 + \tau t) \le e(t_1)^{1-\tau} e(t)^{\tau}$$

$t \to t_2$: $\quad 0 \le e((1-\tau)t_1 + \tau t_2) \le e(t_1)^{1-\tau} \underbrace{e(t_2)^{\tau}}_{=0} = 0$

$\Rightarrow e \equiv 0$ on $[t_1, t_2]$, contradiction. Q.E.D.

<u>Theorem</u> (Smoothness) Suppose $u \in C_1^2(\Omega_T)$ solves the heat equation $u_t - \Delta u = 0$ in Ω_T. Then $u \in C^\infty(\Omega_T)$. Q.E.D.