Chapter 4 Wave Equation

Section 4.1. Method of Separation of Variables
Section 4.2. D'Alembert's Formula. Spherical Means.
Section 4.3. Energy Method

Section 4.1. Method of Separation of Variables

$u = u(x,t)$: $0 < x < l$, $t > 0$, $l > 0$, $c > 0$.

Initial-boundary-value problem of wave equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (0 < x < l,\ t > 0) \\ u(0,t) = 0,\ u(l,t) = 0 & (t > 0) \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) & (0 < x < l) \end{cases}$$

[Homogeneous Dirichlet B.C.]

Try "separated" solution $u(x,t) = X(x)\,T(t)$

$$X T'' = c^2 X'' T. \qquad \frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 & (0 < x < l) \\ X(0) = 0 \quad X(l) = 0 \end{cases}$$

[since $u(0,t) = 0 \Rightarrow X(0)T(t) = 0$.
$u(l,t) = 0 \Rightarrow X(l)T(t) = 0$.]

$$T'' + \lambda c^2 T = 0.$$

Eigenvalues $\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \cdots)$

Eigenfunctions $X_n(x) = \sin\frac{n\pi x}{l} \quad (n = 1, 2, \cdots)$

$$T_n(t) = A_n \cos\frac{n\pi c t}{l} + B_n \sin\frac{n\pi c t}{l} \quad (n = 1, 2, \cdots)$$

Principle of superposition: $u = \sum_{n=1}^{\infty} T_n X_n$.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi c t}{l} + B_n \sin\frac{n\pi c t}{l}\right)\sin\frac{n\pi x}{l}.}$$

Formally,

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-A_n \frac{n\pi c}{l} \sin \frac{n\pi ct}{l} + B_n \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}.$$

Use the initial conditions:

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$$

$$\boxed{A_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx \quad (n=1,2,\cdots)}$$

$$\psi(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$B_n \frac{n\pi c}{l} = \frac{2}{l} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx$$

i.e.,

$$\boxed{B_n = \frac{2}{n\pi c} \int_0^l \psi(x) \sin \frac{n\pi x}{l} dx \quad (n=1,2,\cdots)}$$

Initial-boundary-value problem of wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (0<x<l,\ t>0) \\ u_x(0,t)=0,\ u_x(l,t)=0 & (t>0) \\ u(x,0)=\phi(x),\ u_t(x,0)=\psi(x) & (0<x<l) \end{cases}$$

[Homogeneous Neumann B.C.]

$$\begin{cases} X'' + \lambda X = 0 & (0<x<l) \\ X'(0)=0,\ X'(l)=0 \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n=0,1,2,\cdots.$$

$$X_n(x) = \cos \frac{n\pi x}{l}, \quad n=0,1,2,\cdots.$$

Note $\lambda_0 = 0$, $X_0(x)=1$.

$$T'' + \lambda c^2 T = 0.$$

$$T_0(t) = \frac{A_0}{2} + \frac{B_0}{2} t$$

$$T_n(t) = A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \quad (n=1,2,\cdots)$$

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi c t}{l} + B_n \sin\frac{n\pi c t}{l}\right)\cos\frac{n\pi x}{l}}$$

$$u_t(x,t) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty}\left(-A_n \frac{n\pi c}{l}\sin\frac{n\pi t}{l} + B_n\frac{n\pi c}{l}\cos\frac{n\pi c t}{l}\right)\cos\frac{n\pi x}{l}.$$

Initial conditions:

$$\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}$$

$$\boxed{A_n = \frac{2}{l}\int_0^l \phi(x)\cos\frac{n\pi x}{l}\,dx \quad (n=0,1,2\cdots)}$$

$$\psi(x) = u_t(x,0) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l}\cos\frac{n\pi x}{l}$$

$$\boxed{\begin{aligned} B_0 &= \frac{2}{l}\int_0^l \psi(x)\,dx \\ B_n &= \frac{2}{n\pi c}\int_0^l \psi(x)\cos\frac{n\pi x}{l}\,dx \quad (n=1,2,3,\cdots) \end{aligned}}$$

"Mixed" boundary conditions:

$$u(0,t)=0, \quad u_x(l,t)=0 \qquad (t>0)$$

The eigenvalue problem is

$$\begin{cases} -X'' = \lambda X & (0<x<l) \\ X(0)=0 \quad X'(l)=0 \end{cases}$$

Eigenvalues: $\lambda_n = \left(n+\frac{1}{2}\right)^2\frac{\pi^2}{l^2} \quad (n=0,1,2,\cdots)$

Eigenfunctions: $X_n(x) = \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi}{l}x\right) \quad (n=0,1,2\cdots)$

$$T_n(t) = A_n\cos\left(\left(n+\frac{1}{2}\right)\frac{\pi c t}{l}\right) + B_n \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi c t}{2}\right)$$
$$(n=0,1,2,\cdots)$$

$$u(x,t) = \sum_{n=0}^{\infty}\left[A_n \cos\left((n+\tfrac{1}{2})\frac{\pi c t}{l}\right) + B_n \sin\left((n+\tfrac{1}{2})\frac{\pi c t}{l}\right)\right] \cdot \sin\left((n+\tfrac{1}{2})\frac{\pi x}{l}\right).$$

A_n, B_n $(n=0,1,2,\cdots)$ are determined by the initial conditions $u(x,0)=\phi(x)$ and $u_t(x,0)=\psi(x)$ $(0<x<l)$.

"Mixed" boundary conditions:

$$u_x(0,t)=0, \quad u(l,t)=0, \quad (t>0)$$

$$\begin{cases} -X'' = \lambda X & (0<x<l) \\ X'(0)=0, \quad X(l)=0 \end{cases}$$

$$\lambda_n = (n+\tfrac{1}{2})^2\left(\frac{\pi}{l}\right)^2 \quad (n=0,1,2,\cdots)$$

$$X_n(x) = \cos\left((n+\tfrac{1}{2})\frac{\pi x}{l}\right) \quad (n=0,1,2,\cdots)$$

T_n: same.

$$u(x,t) = \sum_{n=0}^{\infty}\left[A_n \cos\left((n+\tfrac{1}{2})\frac{\pi c t}{l}\right) + B_n \sin\left((n+\tfrac{1}{2})\frac{\pi c t}{l}\right)\right] \cdot \cos\left((n+\tfrac{1}{2})\frac{\pi x}{l}\right).$$

A_n, B_n $(n=0,1,2,\cdots)$ are determined by the initial conditions $u(x,0)=\phi(x)$ and $u_t(x,0)=\psi(x)$ $(0<x<l)$.

Initial-boundary-value problem of wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (\cancel{-l}<x<l, \ t>0) \text{ or } x\in\mathbb{R}, t>0. \\ u(x+l,t) = u(x,t), \ u_t(x+l,t) = u_t(x,t). & (\forall x\in\mathbb{R}^1, \ t>0) \\ u(x,0)=\phi(x), \ u_t(x,0)=\psi(x). \end{cases}$$

[Note: it is here $(-l,l)$ not $(0,l)$]

$\boxed{\text{periodic B.C.}}$

The eigenvalue problem is

$$-X'' = \lambda X \qquad x \in \mathbb{R}, \quad X = X(x) \text{ is } 2l\text{-periodic}$$

Reformulate:
$$\begin{cases} -X'' = \lambda X & (-l < x < l) \\ X(-l) = X(l), & X'(-l) = X'(l). \end{cases}$$

Eigenvalues: $\lambda_0 = 0, \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, \cdots)$

Eigenfunctions: $X_0(x) = 1, \quad X_n(x) = \cos\frac{n\pi x}{l}, \ \sin\frac{n\pi x}{l}$
$(n = 1, 2, \cdots)$

$$T_0(t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t.$$
$$T_n(t) = A_n \cos\frac{n\pi c t}{l} + B_n \sin\frac{n\pi c t}{l} \qquad (n = 1, 2, \cdots)$$

$$\boxed{\begin{aligned} u(x,t) = {} & \frac{1}{2}A_0 + \frac{1}{2}B_0 t \\ & + \sum_{n=1}^{\infty}\left[\left(A_n \cos\frac{n\pi c t}{l} + B_n \sin\frac{n\pi c t}{l}\right)\cos\frac{n\pi x}{l}\right. \\ & \left. + \left(C_n \cos\frac{n\pi c t}{l} + D_n \sin\frac{n\pi c t}{l}\right)\sin\frac{n\pi x}{l}\right] \end{aligned}}$$

Determine all A_n, B_n, C_n, D_n by $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

$$\phi(x) = u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + C_n \sin\frac{n\pi x}{l}\right)$$

$$A_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\cos\frac{n\pi x}{l}\,dx \qquad (n = 0, 1, 2, \cdots)$$

$$C_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\sin\frac{n\pi x}{l}\,dx \qquad (n = 1, 2, \cdots)$$

$$\begin{aligned} u_t(x,t) = {} & \frac{1}{2}B_0 + \sum_{n=1}^{\infty}\left[\left(-A_n\frac{n\pi c}{l}\sin\frac{n\pi c t}{l} + B_n\frac{n\pi c}{l}\cos\frac{n\pi c t}{l}\right)\cdot\right. \\ & \left.\cdot\cos\frac{n\pi x}{l}\right) + \left(-C_n\frac{n\pi c}{l}\sin\frac{n\pi c t}{l} + D_n\frac{n\pi c}{l}\cos\frac{n\pi c t}{l}\right)\sin\frac{n\pi x}{l}\Big] \end{aligned}$$

$$\begin{aligned} \psi(x) = u_t(x,0) = {} & \frac{1}{2}B_0 + \sum_{n=1}^{\infty}\left(B_n\frac{n\pi c}{l}\cos\frac{n\pi x}{l}\right. \\ & \left. + D_n\frac{n\pi c}{l}\sin\frac{n\pi x}{l}\right) \end{aligned}$$

$$B_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} \psi(x)\,dx$$

$$B_n = \frac{1}{n\pi c}\int_{-\ell}^{\ell} \psi(x)\cos\frac{n\pi x}{\ell}\,dx \quad (n=1,2,\cdots)$$

$$D_n = \frac{1}{n\pi c}\int_{-\ell}^{\ell} \psi(x)\sin\frac{n\pi x}{\ell}\,dx \quad (n=1,2,\cdots)$$

Now, let us try to solve a more general initial-boundary-value problem of wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & (0<x<\ell,\ t>0) \\ u(0,t) = g(t),\ u(\ell,t) = h(t) & (t>0) \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) & (0<x<\ell) \end{cases}$$

This is for Dirichlet b.c. Other b.c. can be treated similarly.

First, let us try to get rid of nonzero boundary data. Let $\alpha(t) = g(t)\frac{\ell - t}{\ell} + h(t)\frac{t}{\ell}$.

$$V(x,t) = u(x,t) - \left[g(t)\frac{\ell-t}{\ell} + h(t)\frac{t}{\ell} \right] = u(x,t) - \alpha(t)$$

Then
$$V_t(x,t) = u_t(x,t) - \alpha'(t)$$
$$V_{tt} = u_{tt} - \alpha''(t)$$
$$V_{xx} = u_{xx}.$$

So,
$$\begin{cases} V_{tt} - c^2 V_{xx} = u_{tt} - \alpha''(t) - c^2 u_{xx} = \overbrace{f(x,t) - \alpha''(t)}^{\text{new source term.}} \\ V(0,t) = 0,\ V(\ell,t) = 0 \\ V(x,0) = u(x,0) - \alpha(0) = \phi(x) - \alpha(0) \\ V_t(x,0) = u_t(x,0) - \alpha'(0) = \psi(x) - \alpha'(0) \end{cases}$$

Thus, we need only to ~~con~~sider

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & (0<x<l,\ t>0) \\ u(0,t)=0,\ u(l,t)=0 & (t>0) \\ u(x,0)=\phi(x),\ u_t(x,0)=\psi(x) & (0<x<l) \end{cases}$$

The corresponding eigenvalue problem is

$$\begin{cases} -X'' = \lambda X & (0<x<l) \\ X(0)=0,\ X(l)=0 \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n=1,2,\cdots)$$

$$X_n(x) = \sin\left(\frac{n\pi}{l}x\right) \quad (n=1,2,\cdots)$$

<u>Eigenfunction expansion</u>

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{l}}$$

We need to find all $b_n = b_n(t)$ $(n=1,2,\cdots)$.

Formally, $u_{tt}(x,t) = \sum_{n=1}^{\infty} b_n''(t) \sin\frac{n\pi x}{l}$

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} b_n(t)\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l}$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\frac{n\pi x}{l}. \qquad \boxed{f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx}$$

$u_{tt} - c^2 u_{xx} = f(x,t)$ implies

$$\sum_{n=1}^{\infty}\left(b_n'' + c^2\lambda_n b_n - f_n(t)\right)\sin\frac{n\pi x}{l} = 0 \quad \begin{pmatrix} 0<x<l \\ t>0 \end{pmatrix}$$

Hence, $\quad b_n'' + c^2\lambda_n b_n = f_n(t) \quad (n=1,2,\cdots)$

From the initial condition $u(x,0)=\phi(x)$, we get

$$\phi(x) = \sum_{n=1}^{\infty} b_n(0)\sin\frac{n\pi x}{l}, \qquad \boxed{b_n(0) = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx}$$

Similarly, $u_t(x,0) = \psi(x)$.

$$\psi(x) = \sum_{n=1}^{\infty} b_n'(0)\sin\frac{n\pi x}{l}. \qquad \boxed{b_n'(0) = \frac{2}{l}\int_0^l \psi(x)\sin\frac{n\pi x}{l}\,dx}$$

The equation $b_n'' + c^2 \lambda_n b_n = 0$, i.e., $b_n'' + \left(\frac{c n\pi}{\ell}\right)^2 b_n = 0$, has two linearly indep. solutions

$$\cos\frac{n\pi c}{\ell}t, \quad \sin\frac{n\pi c}{\ell}t.$$

The Wronskian of these two solutions is

$$W_n(t) = \begin{vmatrix} \cos\frac{n\pi c t}{\ell} & \sin\frac{n\pi c}{\ell}t \\ -\frac{n\pi c}{\ell}\sin\frac{n\pi c t}{\ell} & \frac{n\pi c}{\ell}\cos\frac{n\pi c}{\ell}t \end{vmatrix} = \frac{n\pi c}{\ell}.$$

A particular solution to the nomogeneous equation $b_n'' + c^2\lambda_n b_n = f_n(t)$ is given by

$$b_{np}(t) = -\cos\left(\frac{n\pi c t}{\ell}\right)\int_0^t \frac{1}{\frac{n\pi c}{\ell}} f_n(\xi)\sin\frac{n\pi c}{\ell}\xi\, d\xi + \sin\left(\frac{n\pi c t}{\ell}\right)\int_0^t \frac{1}{\frac{n\pi c}{\ell}} f_n(\xi)\cos\frac{n\pi c \xi}{\ell}\, d\xi$$

$$\boxed{b_{np}(t) = -\frac{\ell}{n\pi c}\cos\left(\frac{n\pi c t}{\ell}\right)\int_0^t f_n(\xi)\sin\frac{n\pi c \xi}{\ell}\, d\xi + \frac{\ell}{n\pi c}\sin\left(\frac{n\pi c t}{\ell}\right)\int_0^t f_n(\xi)\cos\frac{n\pi c t}{\ell}\, d\xi}$$

Thus, $b_n(t) = b_{np}(t) + A_n\cos\frac{n\pi c t}{\ell} + B_n\sin\frac{n\pi c t}{\ell}$

A_n, B_n are determined by $b_n(0)$, $b_n'(0)$, given in the boxes above.

$b_{np}(0) = 0$. $\qquad b_n(0) = A_n$

$b_{np}'(0) = 0$. $\qquad b_n'(0) = \frac{n\pi c}{\ell}B_n$.

Summary:

$$\begin{cases} u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{\ell} \quad (0<x<\ell,\ t>0) \\ b_n(t) = b_{n,p}(t) + b_n(0)\cos\frac{n\pi c t}{\ell} + \frac{\ell}{n\pi c}b_n'(0)\sin\frac{n\pi c t}{\ell} \\ b_n(0) = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\frac{n\pi x}{\ell}dx \\ b_n'(0) = \frac{2}{\ell}\int_0^\ell \psi(x)\sin\frac{n\pi x}{\ell}dx \\ b_{np}(t) \text{ is given in the box above.} \end{cases}$$

For space dimension $n \geq 2$, it is more complicated.
Let us try to solve

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in } \Omega \times (0, \infty), \\ u(x,t) = 0 & \text{if } x \in \partial\Omega,\ t > 0. \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), & \text{if } x \in \Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded, smooth domain.
The separation of variables leads to the eigenvalue problem

$$\begin{cases} -\Delta X = \lambda X & X = X(x),\ x \in \Omega. \\ X = 0 \text{ on } \partial\Omega \end{cases}$$

Eigenvalues: $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \to \infty$.
Eigenfunctions $\varphi_k = \varphi_k(x)$. $(k = 1, 2, 3, \cdots)$

$\{\varphi_k\}_{k=1}^{\infty}$ is a complete, orthonormal basis of $L^2(\Omega)$: ① $\langle \varphi_j, \varphi_k \rangle$

$$= \int_\Omega \varphi_j(x)\varphi_k(x)\,dx = \delta_{jk} = \begin{cases} 1 & \text{if } j = k. \\ 0 & \text{if } j \neq k. \end{cases}$$

② For any $f \in L^2(\Omega)$, we have $f = \sum_{k=1}^{\infty} f_k \varphi_k$,
$(f_k = \langle f, \varphi_k \rangle,\ k = 1, 2, \cdots)$.

$$T_n(t) = A_n \cos\left(\cancel{\frac{n\pi c}{l}}\sqrt{\lambda_n}c\, t\right) + B_n \sin\left(\cancel{\frac{n\pi c}{l}}\sqrt{\lambda_n}c\, t\right) \quad (n = 1, 2, \cdots)$$

So, $$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct)\right)\varphi_n(x)}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-A_n\sqrt{\lambda_n}c \sin(\sqrt{\lambda_n}ct) + B_n\sqrt{\lambda_n}c\cos(\sqrt{\lambda_n}ct)\right)\varphi_n(x)$$

$$\phi(x) = \sum_{n=1}^{\infty} \cancel{B_n\sqrt{\lambda_n}}\, A_n \varphi_n(x). \qquad \boxed{\begin{array}{l} A_n = \int_\Omega \phi(x)\varphi_n(x)\,dx \\ (n = 1, 2, \cdots) \end{array}}$$

$$\psi(x) = \sum_{n=1}^{\infty} B_n\sqrt{\lambda_n}c\,\varphi_n(x) \qquad \boxed{\begin{array}{l} B_n = \frac{1}{\sqrt{\lambda_n}c}\int_\Omega \psi(x)\varphi_n(x)\,dx \\ (n = 1, 2, \cdots) \end{array}}$$