Section 4.2 D'Alembert's Formula. Spherical Means

We first consider $u = u(x,t)$ for $-\infty < x < \infty$, $t > 0$.
Suppose $u = u(x,t)$ satisfies the one-dimensional wave equation ($c > 0$ is a constant):

$$u_{tt} - c^2 u_{xx} = 0$$

Let $\begin{cases} \xi = x + ct, \\ \eta = x - ct. \end{cases}$ Thus $\begin{cases} x = \frac{\xi+\eta}{2} \\ t = \frac{\xi-\eta}{2c} \end{cases}$

Let $\hat{u}(\xi,\eta) = u(x,t) = u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right)$.

$$\hat{u}_\xi = u_x x_\xi + u_t t_\xi = \frac{1}{2} u_x + \frac{1}{2c} u_t$$

$$\begin{aligned} \hat{u}_{\xi\eta} &= \frac{1}{2} u_{xx} x_\eta + \frac{1}{2} u_{xt} t_\eta + \frac{1}{2c} u_{tx} x_\eta + \frac{1}{2c} u_{tt} t_\eta \\ &= \frac{1}{2} u_{xx} \cdot \frac{1}{2} + \frac{1}{2} u_{xt}\left(-\frac{1}{2c}\right) + \frac{1}{2c} u_{tx} \frac{1}{2} + \frac{1}{2c} u_{tt}\left(-\frac{1}{2c}\right) \\ &= \frac{1}{4}\left(u_{xx} - \frac{1}{c^2} u_{tt}\right) = 0 \end{aligned}$$

$$\boxed{\hat{u}_{\xi\eta} = 0} \Longrightarrow \hat{u}_\xi = g(\xi). \quad \boxed{\hat{u}(\xi,\eta) = F(\xi) + G(\eta)}$$

$$F(\xi) = \int g(\xi)\,d\xi$$

$$\boxed{u(x,t) = F(x+ct) + G(x-ct)}$$

This is the general solution to the one-dimensional wave equation, where F, G are two one-variable (smooth) functions.

Consider now the initial-value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (-\infty < x < \infty, \ t > 0) \\ u(x,0) = \phi(x), \ u_t(x,0) = \psi(x), & (-\infty < x < \infty). \end{cases}$$

Since $u(x,t) = F(x+ct) + G(x-ct)$,
we have $u_t(x,t) = cF'(x+ct) - cG'(x-ct)$.

$u(x,0) = \phi(x).$

$\phi(x) = F(x) + G(x) \Longrightarrow \phi'(x) = F'(x) + G'(x)$

$u_t(x,0) = \psi(x)$

$\psi(x) = cF'(x) - cG'(x) \Longrightarrow \frac{\psi(x)}{c} = F'(x) - G'(x)$

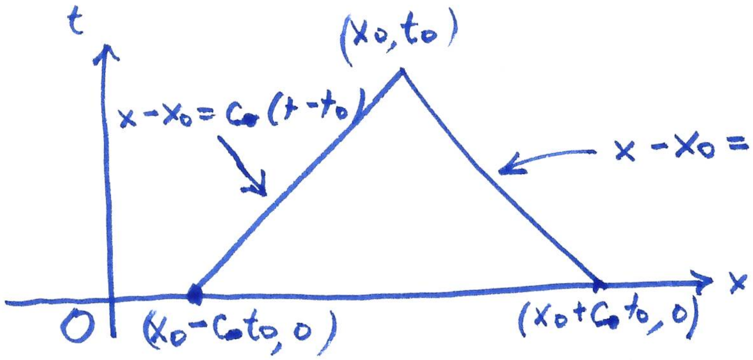
Hence,
$$F'(x) = \frac{1}{2}\phi'(x) + \frac{1}{2c}\psi(x).$$
$$F(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds + C_1$$
$$G'(x) = \frac{1}{2}\phi'(x) - \frac{1}{2c}\psi(x)$$
$$G(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds + C_2$$

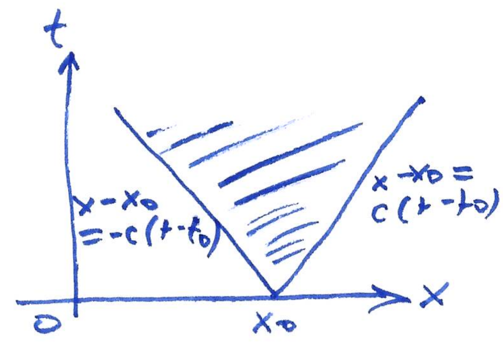
Since, $F(x) + G(x) = \phi(x)$, we have $C_1 + C_2 = 0$.

Thus,
$$\begin{aligned} u(x,t) &= F(x+ct) + G(x-ct) \\ &= \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi(s)\,ds + C_1 \\ &\quad + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct}\psi(s)\,ds + C_2 \end{aligned}$$

$$\boxed{u(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds}$$

This is d'Alembert's formula.





The value $u(x_0, t_0)$ is determined by the ϕ and ψ values on $[x_0 - c_0t_0, x_0 + ct_0]$. This interval is called <u>the domain of dependence</u> for (x_0, t_0).

Any $x_0 \in \mathbb{R}$ corresponds to a wedge-shaped region in the xt-plane, called <u>the range of influence of x_0</u>.

Method of spherical means for wave equation in high space dimension $n \geq 2$.

Let $u = u(x)$ $(x \in \mathbb{R}^n)$ be a smooth function. For any $x \in \mathbb{R}^n$ and $r > 0$, define

$$U(x,r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y) = \fint_{\partial B(x,r)} u(y)\, dS(y),$$

where $|\partial B(x,r)|$ is the surface area of sphere $\partial B(x,r)$:

$|\partial B(x,r)| = \omega_n r^{n-1}$, ω_n = area of unit sphere in $\mathbb{R}^n$.

Call $U = U(x,r)$ the spherical mean of $u = u(x)$ at x, r. We have by a change of variable

$$U(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} u(x + r\xi)\, dS(\xi).$$

Thus,

$$U_r(x,r) = \frac{1}{\omega_n} \int_{|\xi|=1} (\nabla u)(x + r\xi) \cdot \xi \, dS(\xi)$$

$$\overset{y = x + r\xi}{=} \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r}\, dS(y)$$

$$= \frac{r}{n\,|B(x,r)|} \int_{\partial B(x,r)} \frac{\partial u(y)}{\partial \nu}\, dS(y)$$

$\boxed{|B(x,r)| = \text{vol. of } B(x,r)}$

$$\overset{\text{Divergence Thm}}{=} \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\, dy$$

Clearly,

$$U(x,0) = u(x)$$

$$U_r(x,r)\Big|_{r=0} = \lim_{r \to 0} U_r(x,r) = 0.$$

Re do the calculations.

$$U_r(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} \sum_{j=1}^{n} u_{x_j}(x+r\xi)\,\xi_j\, dS(\xi)$$

$$= \frac{r}{\omega_n}\int_{|\xi|<1} \Delta_x u(x+r\xi)\,d\xi$$

$$= \frac{r}{\omega_n}\Delta_x \int_{|\xi|<1} u(x+r\xi)\,d\xi$$

$$= \frac{r^{1-n}}{\omega_n}\Delta_x \int_{B(x,r)} u(y)\,dy$$

$$= \frac{r^{1-n}}{\omega_n}\Delta_x \int_0^r d\rho \int_{\partial B(x,\rho)} u(y)\,dS(y)$$

$$= \frac{r^{1-n}}{\cancel{\omega_n}}\Delta_x \int_0^r \rho^{n-1}\,U(x,\rho)\,d\rho$$

So, $\quad r^{n-1}U_r(x,r) = \Delta_x \int_0^r \rho^{n-1} U(x,\rho)\,d\rho$

$\frac{d}{dr}$: $\quad \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}U(x,r)\right) = \Delta_x\, r^{n-1}\,U(x,r).$

Hence $\quad \boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)U(x,r) = \Delta_x U(x,r)}$

— Darboux's equation.

$$U(x,0) = u(x), \quad \left(\frac{\partial}{\partial r}U(x,r)\right)_{r=0} = 0.$$

Now, consider $u = u(x,t)$ $(x\in\mathbb{R}^n,\ t>0)$:

$$\begin{cases} u_{tt} - c^2\Delta u = 0 \\ u(x,0) = g(x),\ u_t(x,0) = h(x), \quad x\in\mathbb{R}^n \end{cases}$$

Let

$$U(x,r,t) = \fint_{\partial B(x,r)} u(y,t)\,dS(y),$$

$$G(x,r) = \fint_{\partial B(x,r)} g(y)\,dS(y),$$

$$H(x,r) = \fint_{\partial B(x,r)} h(y)\,dS(y).$$

Note that: $u(x,t) = U(x,0,t)$ (recovering $u(x,t)$)

Darboux's equation

$$\Delta_x U(x,r,t) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) U(x,r,t).$$

On the other hand,

$$\Delta_x U(x,r,t) = \Delta_x\left(\frac{1}{\omega_n}\int_{|\xi|=1} u(x+r\xi,t)\,dS(\xi)\right)$$

$$= \frac{1}{\omega_n}\int_{|\xi|=1} \Delta_x u(x+r\xi,t)\,dS(\xi)$$

$$= \frac{1}{\omega_n}\int_{|\xi|=1} \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u(x+r\xi,t)\,dS(\xi)$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{\omega_n}\int_{|\xi|=1} u(x+r\xi,t)\,dS(\xi)\right)$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} U(x,r,t)$$

Note: $\Delta_x w(|x|) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) w(r)$ $\left(\begin{array}{l} w(r) = w(|x|) \\ r = |x|. \end{array}\right)$

Hence, for each fixed $x \in \mathbb{R}^n$,

$$\boxed{U_{tt} = c^2\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) U}$$

— Euler-Poisson-Darboux equation.

Initial conditions:

$$\boxed{\begin{array}{l} U(x,r,t=0) = G(x,r) \\ U_t(x,r,t=0) = H(x,r) \end{array}}$$

Solve
$$\begin{cases} U_{tt} = c^2\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)U \\ U(x,r,t=0)=G(x,r), \quad U_t(x,r,t=0)=H(x,r). \end{cases}$$

$\boxed{\text{for } n=3}$

$$\begin{cases} \frac{\partial^2}{\partial t^2}(rU) = c^2 \frac{\partial^2}{\partial r^2}(rU) \\ (rU)(t=0)=rG, \quad \frac{\partial}{\partial t}(rU)(t=0)=rH. \end{cases}$$

Apply d'Alembert's formula:

$$rU(x,r,t)=\frac{1}{2}\left[(r+ct)G(x,r+ct)+(r-ct)G(x,r-ct)\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} sH(x,s)\,ds$$

Using the radial symmetry, we have

$$U(x,r,t)=\frac{1}{2r}\left[(ct+r)G(x,ct+r)-(ct-r)G(x,ct-r)\right] + \frac{1}{2rc}\int_{ct-r}^{ct+r} sH(x,s)\,ds$$

Now, $r\to 0^+$. note: $U(x,0,t)=u(x,t)$.

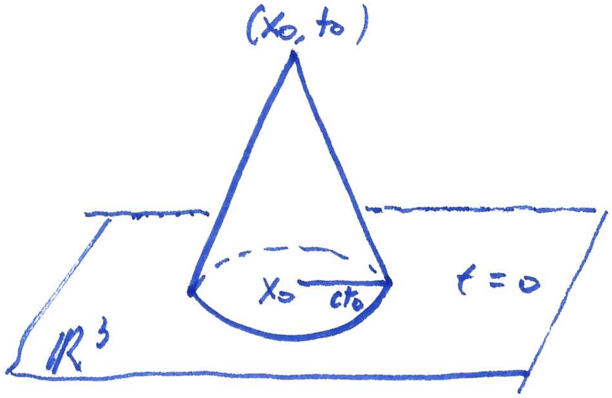
$$u(x,t) = \frac{\partial}{\partial \tau}\left(\tau G(x,\tau)\right)\Big|_{\tau=ct} + tH(x,ct)$$
$$= \frac{\partial}{\partial t}\left(tG(x,ct)\right) + tH(x,ct)$$

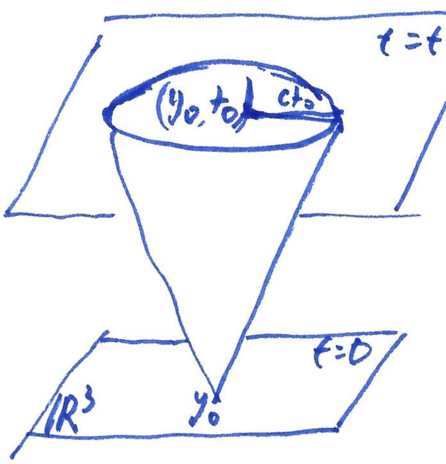
Finally, for $\boxed{n=3}$.

$$\boxed{u(x,t)=\frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{\partial B(x,ct)} g(y)\,dS(y)\right) + \frac{1}{4\pi c^2 t}\int_{\partial B(x,ct)} h(y)\,dS(y)}$$

Remarks ① Solution existence and uniqueness by the solution formula.

② $u(x,t)$ depends on g, h on a sphere (surface, not solid region)





<u>The domain of dependence</u> for $u(x_0,t_0)$ is the sphere $\partial B(x_0, ct_0)$ in $\mathbb{R}^n$.

The values of $g(y_0)$, $h(y_0)$ only affect u values on ~~the~~ sphere $\partial B(y_0, ct_0)$ at time $t_0>0$.

<u>Huygen's Principle.</u>
A disturbance originating at x propagetes along a sharp wavefron – sphere.
[Only for $n \geq 3$ and n. odd.]

Now, $\boxed{n=2}$. Hadamard's method of descent

$u = u(x_1, x_2, t)$. $\quad g = g(x_1, x_2)$, $\quad h = h(x_1,x_2)$

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty) \\ u = g, \ u_t = h & \text{at } t=0 \end{cases}$$

This is true when $\mathbb{R}^2$ is replaced by $\mathbb{R}^3$. Extend (x_1, x_2) to $(x_1, x_2, 0)$. So the sphere $\partial B(x, ct)$ is the set of $y = (y_1, y_2, y_3)$ such that

$$|y - x| = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2 + y_3^2} = ct$$

On this surface $g(y)$ ~~$= g(\ \ )$~~ $= g(y_1, y_2, y_3) = g(y_1, y_2, -y_3)$ $= g(y_1, y_2)$. So, let us consider $dS(y)$ for the upper half sphere $y_3 = \sqrt{(ct)^2 - (y_1-x_1)^2 - (y_2-x_2)^2}$.

$$dS(y) = \sqrt{1 + \left(\frac{\partial y_3}{\partial y_1}\right)^2 + \left(\frac{\partial y_3}{\partial y_2}\right)^2}\, dy_1 dy_2 = \frac{ct}{|y_3|} dy_1 dy_2 .$$

Now, by the formula for $n=3$, we obtain

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi c} \int_{(y_1-x_1)^2+(y_2-x_2)^2<(ct)^2} \frac{g(y_1,y_2)}{\sqrt{(ct)^2-(y_1-x_1)^2-(y_2-x_2)^2}} dy_1 dy_2 \right)$$

$$+ \frac{1}{2\pi c} \int_{(y_1-x_1)^2+(y_2-x_2)^2<(ct)^2} \frac{h(y_1,y_2)}{\sqrt{(ct)^2-(y_1-x_1)^2-(y_2-x_2)^2}} dy_1 dy_2 .$$

Using $x=(x_1,x_2)$, $y=(y_1,y_2)$

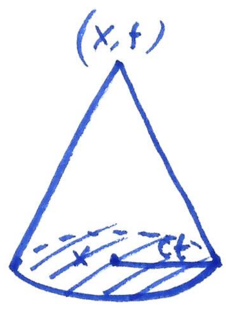
$$B(x,ct) = \{ y=(y_1,y_2) \in \mathbb{R}^2 : (y_1-x_1)^2+(y_2-x_2)<(ct)^2 \},$$

we have for $\boxed{n=2}$

that

$$\boxed{u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi c} \int_{B(x,ct)} \frac{g(y)\,dy}{\sqrt{(ct)^2-|y-x|^2}}\right) + \frac{1}{2\pi c}\int_{B(x,ct)} \frac{h(y)}{\sqrt{(ct)^2-|y-x|^2}} dy}$$

<u>The domain of dependence</u> for $u(x,t)$ is now the disk in $\mathbb{R}^2$ with center x and radius ct.



We omit formulas for general $n \geq 4$: two cases n: even, n: odd. See e.g., Evan's PDE book.

Duhamel's Principle for nonhomogeneous wave equation

$$\begin{cases} u_{tt} - c^2 \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = 0 \text{ and } u_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

Define $u_s = u(x,t,s)$ by

$$u_{s,tt} - c^2 \Delta u_s = 0 \quad \mathbb{R}^n \times (0,\infty),$$
$$u_s = 0, \ u_{st} = f(\cdot, s) \quad \mathbb{R}^n \times \{t = s\}.$$

Then $$u(x,t) = \int_0^t u(x,t,s)\, ds.$$