Section 4.3 Energy Method / Fourier Transforms

Let $u = u(x,t)$ solve $u_{tt} - c^2 \Delta u = 0$. Assume $u(x,t) = 0$ for any $t \geq 0$ and x outside a finite ball centered at 0 with radius depending on t. This is true if the initial data have compact supports.

Define the energy

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^n} \left(u_t^2 + c^2 |\nabla u|^2 \right) dx.$$

Then

$$E'(t) = \int_{\mathbb{R}^n} \left(u_t u_{tt} + c^2 \sum_{j=1}^n u_{x_j} u_{x_j t} \right) dx$$

$$= \int_{\mathbb{R}^n} \left(u_t u_{tt} - c^2 \sum_{j=1}^n u_{x_j x_j} u_t \right) dx$$

$$= \int_{\mathbb{R}^n} u_t \left(u_{tt} - c^2 \Delta u \right) dx = 0.$$

So, $E(t) = \text{const}$. <u>Conservation of energy</u>!

<u>Effects of lower order terms</u> Just one-dimensional (for simplicity)

<u>Example</u> $u = u(x,t)$, $(x \in \mathbb{R}, t > 0)$.
(Dispersion) $u_{tt} - u_{xx} + \lambda u = 0$.

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left(u_t^2 + u_x^2 + \lambda u^2 \right) dx$$

$$E'(t) = \int_{-\infty}^{\infty} \left(u_t u_{tt} + u_x u_{xt} + \lambda u u_t \right) dx$$

$$= \int_{-\infty}^{\infty} (u_t u_{tt} - u_{xx} u_t + \lambda u u_t)\,dx$$ [Assume $u(x,t)$ has compact support for each t.]

$$= \int_{-\infty}^{\infty} u_t (u_{tt} - u_{xx} + \lambda u)\,dx$$

$$= 0$$

So, $E(t) = \text{const}$.

Do more now. Try $u(x,t) = U(kx - \omega t)$

[wave propagation with constant velocity $c = \frac{\omega}{k}$.]

$$(\omega^2 - k^2) U'' + \lambda U = 0$$ [Just plug into $u_{tt} - u_{xx} + \lambda u = 0$].

For U to be bounded, we need $\lambda^{-1}(\omega^2 - k^2) > 0$.

Then, U is a combination of cosine and sine functions of $kx - \omega t$. So, let's try

$$u(x,t) = A e^{i(kx - \omega t)}$$

Now,

$$u_t = A e^{i(kx-\omega t)} \cdot (-i\omega)$$

$$u_{tt} = A e^{i(kx-\omega t)} (-i\omega)^2 = -\omega^2 u$$

$$u_x = A e^{i(kx-\omega t)} (ik)$$

$$u_{xx} = A e^{i(kx-\omega t)} (ik)^2 = -k^2 u$$

Plug into $u_{tt} - u_{xx} + \lambda u = 0$

to get

$$-\omega^2 + k^2 + \lambda = 0$$

$$\boxed{\omega(k) = \pm\sqrt{k^2 + \lambda}}$$ (for $k^2 + \lambda \geq 0$)

Called <u>the dispersion relation</u>.

$$c = c(k) = \frac{\omega(k)}{k} \quad : \text{ phase velocity}$$

Now, $u(x,t) = \sum_{j=1}^{N} A_j e^{i(k_j x - \omega_j t)}$

If ~~#~~ each $A_j e^{i(k_j x - \omega_j t)}$ solves the equation then $\omega_j = \omega_j(k_j) = \pm\sqrt{k_j^2 + \lambda}$

So, different speeds $c_j = c_j(k_j)$. — $\boxed{\text{dispersion}}$.

Example (Dissipation) $u_{tt} - u_{xx} + \alpha u_t + \beta u_x + \gamma u = 0$

α, β, γ: constants. $\alpha > 0$.

Let $\xi = x+t$, $\eta = x-t$. Then

$$u_{\xi\eta} - \frac{\alpha+\beta}{4} u_\xi + \frac{\alpha-\beta}{4} u_\eta - \frac{\gamma}{4} u = 0$$

Set $\quad w(\xi,\eta) = u(\xi,\eta)\, e^{\frac{\alpha-\beta}{4}\xi - \frac{\alpha+\beta}{4}\eta}$

Then.

$$w_{\xi\eta} + \frac{\lambda}{4} w = 0. \qquad \left(\lambda = \frac{\alpha^2 - \beta^2 - 4\gamma}{4}\right)$$

Back to (x,t):

$$u(x,t) = w(x,t)\, e^{\frac{\beta}{2}x - \frac{\alpha}{2}t}$$

w: conserves energy. u: energy decays due $e^{-\frac{\alpha}{2}t}$. $(\alpha > 0)$.

True?. First, define

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (u_t^2 + u_x^2 + \gamma u^2)\,dx.$$

Assume compact support of initial data, we have

$$E'(t) = \int_{-\infty}^{\infty} (u_t u_{tt} + u_x u_{xt} + \gamma u u_t)\,dx$$

$$= \int_{-\infty}^{\infty} (u_t u_{tt} - u_{xx} u_t + \gamma u u_t)\,dx$$

$$= \int_{-\infty}^{\infty} u_t (u_{tt} - u_{xx} + \gamma u)\,dx$$

$$= -\alpha \int_{-\infty}^{\infty} u_t^2\,dx \le 0.$$

So. $E(t) \downarrow$. (as $t \uparrow$).

Fourier Transforms

$$\begin{cases} u_{tt} - c^2 \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^n \end{cases}$$

$$\hat{u}(\xi,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x,t) e^{-i x\cdot\xi} dx$$

$$\hat{g}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} g(x) e^{-i x\cdot\xi} dx$$

$$\hat{h}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} h(x) e^{-i x\cdot\xi} dx$$

Note. $\widehat{\Delta u}(\xi,t) = -|\xi|^2 \hat{u}(\xi,t)$.

Thus, for each $\xi \in \mathbb{R}^n$, we have

$$\begin{cases} \hat{u}_{tt} + c^2|\xi|^2 \hat{u} = \hat{f} \\ \hat{u}(\xi,0) = \hat{g}(\xi), \quad \hat{u}_t(\xi,0) = \hat{h}(\xi). \end{cases}$$

$\cos(c|\xi|t)$, $\sin(c|\xi|t)$ are two linearly indep solutions to the homogeneous equation (ODE).

$$\hat{u}_{tt} + c^2|\xi|^2\hat{u} = 0.$$

The Wronskian is

$$W(\xi,t) = \begin{vmatrix} \cos(c|\xi|t) & \sin(c|\xi|t) \\ -c|\xi|\sin(c|\xi|t) & c|\xi|\cos(c|\xi|t) \end{vmatrix} = c|\xi|.$$

A particular solution $\hat{u}_p = \hat{u}_p(\xi,t)$ for the nonhomogneous equation $\hat{u}_{tt} + c^2|\xi|^2\hat{u} = \hat{f}$ is (for $|\xi| \neq 0$)

$$\hat{u}_p(\xi,t) = -\frac{\cos(c|\xi|t)}{c|\xi|}\int_0^t \sin(c|\xi|s)\hat{f}(\xi,s)\,ds + \frac{\sin(c|\xi|t)}{c|\xi|}\int_0^t \cos(c|\xi|s)\hat{f}(\xi,s)\,ds$$

It is easy to verify that $\hat{u}_p(\xi,0)=0$, $\hat{u}_{pt}(\xi,0)=0$.

The general solution is

$$\hat{u}(\xi,t) = \hat{u}_p(\xi,t) + C_1(\xi)\cos(c|\xi|t) + C_2(\xi)\sin(c|\xi|t).$$

Initial conditions determine $C_1(\xi)$ and $C_2(\xi)$. Finally,

$$\hat{u}(\xi,t) = \hat{u}_p(\xi,t) + \hat{g}(\xi)\cos(c|\xi|t) + \frac{\hat{h}(\xi)}{c|\xi|}\sin(c|\xi|t).$$

$$= \hat{g}(\xi)\cos(c|\xi|t) + \frac{\hat{h}(\xi)}{c|\xi|}\sin(c|\xi|t)$$

$$+\frac{1}{c|\xi|}\int_0^t \left[\cos(c|\xi|s)\sin(c|\xi|t) - \sin(c|\xi|s)\cos(c|\xi|t)\right]\hat{f}(\xi,s)\,ds$$

$$= \hat{g}(\xi)\cos(c|\xi|t) + \frac{\hat{h}(\xi)}{c|\xi|}\sin(c|\xi|t)$$

$$+ \frac{1}{c|\xi|}\int_0^t \sin(c|\xi|(t-s))\hat{f}(\xi,s)\,ds$$

$$\boxed{\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{g}(\xi)\cos(c|\xi|t)\,d\xi \\ &+ \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{\hat{h}(\xi)}{c|\xi|}\sin(c|\xi|t)\,d\xi \\ &+ \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{c|\xi|}\left(\int_0^t \sin(c|\xi|(t-s))\hat{f}(\xi,s)\,ds\right)d\xi. \end{aligned}}$$

Valid for e.g., smooth, compactly supported data, f, g, h.