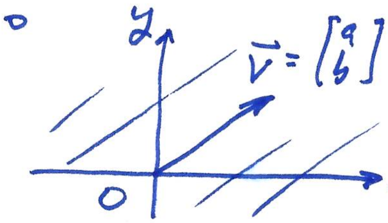
Chapter 5 First-Order PDE

Section 5.1 Method of Characteristics / Complete Integrals

Section 5.2 Hamilton-Jacobi Equations

Section 5.3 Hyperbolic Conservation Laws

Section 5.1 Method of Characteristics / Complete Integrals



Transport equation $a u_x + b u_y = 0$

$a, b = \text{const.}$ $u = u(x, y)$, $x, y \in \mathbb{R}$.

Observe: $\nabla u \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$

$u = \text{const.}$ along the direction $\begin{bmatrix} a \\ b \end{bmatrix}$.

E.g., $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\nabla u \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_x = 0 \Longrightarrow u = u(y)$.

$u = \text{const.}$ on lines $// \vec{v}$ $\Longleftrightarrow$ lines $//$ $y = \frac{b}{a} x$, i.e., $bx - ay = 0$.

$\Longleftrightarrow$ lines: $bx - ay = \text{const.}$

So, $u(x, y) = \text{const.}$ on $bx - ay = \text{const.}$

u only depends on $bx - ay$

$$\boxed{u(x, y) = f(bx - ay), \quad f: \text{single-variable } C^1\text{-function.}}$$

check: $u_x = f'(bx - ay)\, b$

$u_y = f'(bx - ay)(-a)$

$a u_x + b u_y = abf' - abf' = 0$.

A different way to look at this equation:

$$u_t + b \cdot \nabla u = 0 \qquad (x \in \mathbb{R}^n,\ t > 0)$$

Here $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ is a constant vector.

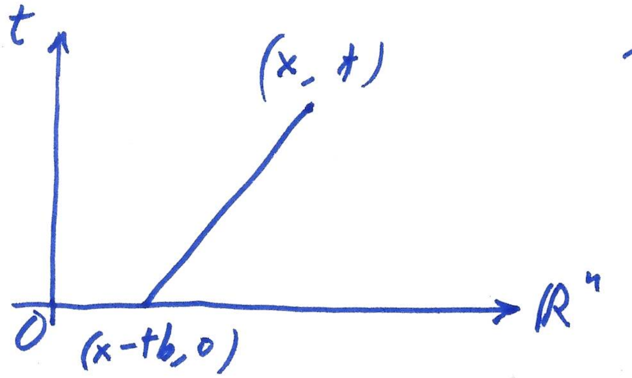
Fix (x,t). Define $z = z(s) = u(x+sb,\ t+s)$. Then

$$\frac{dz}{ds} = \nabla u(x+sb, t+s) \cdot b + u_t(x+sb, t+s) = 0$$

So, $z(s) = \text{const.}$ (indep. on s).

i.e., $u(x+sb,\ t+s) = \text{const.}$

$\gamma = \{ (x+sb,\ t+s) : s \in \mathbb{R} \}$ is a curve (a line in this case). It intersects $\mathbb{R}^n$ at $(x-tb, 0)$ (This is obtained by choosing $s = -t$: $(x+sb,\ t+s) = (x-bt,\ 0)$.



Thus, $\underbrace{u(x,t) = u(x-tb, 0)}_{\text{Call this } f(x-tb)}$.

So, the solution is

$$\boxed{u(x,t) = f(x-tb)}$$

f: single-variable function. C^1-function.

$$\begin{cases} x = x_0 + sb \\ t = t_0 + s \end{cases}$$ characteristic lines

$$\begin{cases} \frac{dx}{ds} = b \\ \frac{dt}{ds} = 1. \end{cases}$$

Back to $a u_x + b u_y = 0$.

$$\begin{cases} x = x_0 + as \\ y = y_0 + bs. \end{cases}$$ characteristic lines

$bx - ay = bx_0 - ay_0 = \text{const.}$

(indep. of s)

$u(x,y) = f(bx - ay)$.

Initial-value problem of linear transport equation (with constant coefficients)

$$\begin{cases} u_t + b\cdot\nabla u = 0 & (x,t)\in\mathbb{R}^n\times(0,\infty) \\ u(x,0) = u_0(x) & x\in\mathbb{R}^n \end{cases}$$

$$\begin{cases} \dfrac{dx}{ds} = b \\ \dfrac{dt}{ds} = 1 \end{cases} \Longrightarrow \text{characteristics} \quad \begin{cases} x = x_0 + bs \\ t = t_0 + s \end{cases} \quad \begin{matrix} (s\in\mathbb{R}) \\ (\text{or } s\geq -t_0) \end{matrix}$$

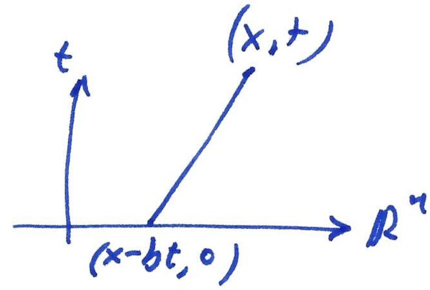
$x - bt = x_0 - bt_0 = \text{const.}$ (indep. of s).

$u = \text{const.}$ along $x - bt = \text{const.}$

So, $u(x,t) = f(x-bt)$.

Set $t=0$: $u_0(x) = f(x)$. So,

$$\boxed{u(x,t) = u_0(x-bt)}$$



Nonhomogeneous Problems

$$\begin{cases} u_t + b\cdot\nabla u = f & \text{in } \mathbb{R}^n\times(0,\infty) \\ \qquad\quad u = g & \text{on } \mathbb{R}^n\times\{t=0\} \end{cases}$$

Set $z(s) = u(x+sb, t+s)$ $(s\in\mathbb{R})$

Then
$$\dot z(s) = \frac{dz}{ds} = \nabla u(x+sb, t+s)\cdot b + u_t(x+sb, t+s)$$
$$= f(x+sb, t+s)$$

Thus
$$u(x,t) - g(x-tb) = z(0) - z(-t) = \int_{-t}^{0} \dot z(s)\,ds$$
$$= \int_{-t}^{0} f(x+sb, t+s)\,ds$$
$$= \int_0^t f(x+(s-t)b, s)\,ds.$$

$$\boxed{u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)\,ds \quad (x\in\mathbb{R}^n,\ t\geq 0)}$$

First-order linear equation, variable coefficients

$$b(x)\cdot\nabla u(x) + c(x)\,u(x) = 0 \qquad x\in\Omega\subseteq\mathbb{R}^n$$

$$b(x) = \begin{bmatrix} b_1(x) \\ \vdots \\ b_n(x)\end{bmatrix}, \qquad \nabla u(x) = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n}\end{bmatrix}$$

Consider curves $x = x(s)$. Let $z = z(s) = u(x(s))$

Then $\dot z(s) = \nabla u(x(s))\cdot\dot x(s)$

Compare with $b\cdot\nabla u + cu = 0$

$$\begin{cases} \dot x(s) = b(x(s)) \\ \dot z(s) = -c(x(s))\,z \end{cases}$$

These are the characteristic equations: ODEs.

<u>Example</u>
$$\begin{cases} (1+x^2)u_x - 2x\,u_y = 0 & (x, y\in\mathbb{R}) \\ u(0,y) = \sin y & (y\in\mathbb{R}) \end{cases}$$

$z = z(s) = u(x(s), y(s))$

$\dot z = u_x\,\dot x + u_y\,\dot y = 0$. So,
$$\begin{cases} \dot x = 1+x^2 \\ \dot y = -2x \\ \dot z = 0. \end{cases}$$

$$\frac{dy}{dx} = \frac{\dot y}{\dot x} = -\frac{2x}{1+x^2}$$

$$y = -\ln(1+x^2) + C. \qquad (*)$$

$\dot z = 0$ means on this curve $u = \text{const}$. So,

$$u(x,y) = u(x, C - \ln(1+x^2))$$
$$\underset{\substack{\uparrow \\ x=0}}{=} u(0, C) \underset{\substack{\uparrow \\ \text{by } (*)}}{=} u(0, y + \ln(1+x^2))$$

Thus, $\boxed{u(x,y) = f(y+\ln(1+x^2))}$ (f: a single-variable C^1-function.)

General sol'n of the equation — involving an arbitrary function (not a const.)

Now, let us find f.

$$\sin y = u(0, y) = f(y)$$

Finally,

$$u(x,y) = \sin\left(y + \ln(1+x^2)\right).$$

Check it!

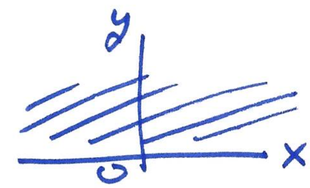
Example 2
$$\begin{cases} x u_y - y u_x = u & (x \geq 0,\ y \geq 0) \\ u(x,0) = g(x), & (x \geq 0) \end{cases}$$

$$z(s) = u(x(s), y(s))$$
$$\dot z = u_x \dot x + u_y \dot y \quad \Longrightarrow \quad \begin{cases} \dot x = -y \\ \dot y = x \\ \dot z = z \end{cases}$$

$$\begin{cases} x(s) = C\cos s \\ y(s) = C\sin s \\ z(s) = De^s \end{cases} \qquad \begin{aligned} & x^2 + y^2 = C^2 \\ & s = \arctan\frac{y}{x} \end{aligned}$$

$$u(x(s), y(s)) = u(C\cos s, C\sin s)$$
$$s=0: \quad u(x(0), y(0)) = u(C, 0) = g(C)$$
$$z(s) = u(x(s), y(s))$$
$$z(0) = u(x(0), y(0)) = g(C) \qquad \text{So, } D = g(C)$$
$$u(x,y) = z(s) = De^s = g(C)e^s$$
$$= g\left(\sqrt{x^2+y^2}\right) e^{\arctan\frac{y}{x}}.$$

Some nonlinear problem.



Example 3
$$\begin{cases} u_x + u_y = u^2 & x \in \mathbb{R},\ y > 0 \\ u(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

$$z(s) = u(x(s), y(s))$$

$$\dot{z}(s) = u_x \dot{x} + u_y \dot{y} = z^2$$

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 1 \\ \dot{z} = z^2 \end{cases}$$

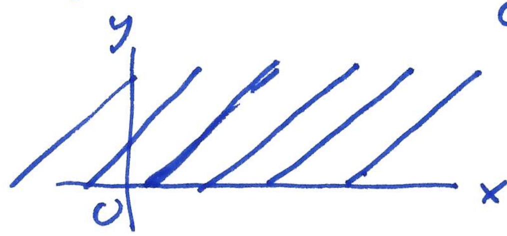
These are called characteristic equations.

$\begin{cases} \dot{x} = 1 \\ \dot{y} = 1 \end{cases}$ determine characteristics (or characteristic curves) — lines in this case.

It's tricky in choosing s.

$[x(s) = s = x]$

Here $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = 1$ So, $y = x - x_0$ characteristics



$$\dot{z} = z^2 \Longrightarrow z = \frac{z_0}{1 - s z_0}$$

But $s = x$ So,

$$z = \frac{z_0}{1 - z_0 x}$$

$$z(0) = z_0 = u(x(0), y(0)) = u(x_0, 0) = g(x_0)$$

So, $z(x) = \frac{g(x_0)}{1 - g(x_0)x}$

Finally, $u(x,y) = z(s) = \frac{g(x_0)}{1 - g(x_0)x} = \frac{g(x-y)}{1 - x\,g(x-y)}$

provided that the denominator is nonzero.

Fully nonlinear: $F(x, u, \nabla u) = 0.$ $x \in \Omega,$

$u = g$ $x \in \Gamma \subseteq \partial\Omega.$

The resulting set of characteristic equations are more complicated.

Section 5.2 Hamilton-Jacobi Equations

$$\begin{cases} u_t + H(\nabla u, x) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

$u = u(x,t)$, $(x \in \mathbb{R}^n,\ t>0)$

characteristic equations

$$\begin{cases} \dot{x} = \cancel{B}\, \nabla_p H(p,x) \\ \dot{p} = -\nabla_x H(p,x) \end{cases}$$

The calculus of variations

Consider a smooth function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Call it a Lagrangian. Write

$$L = L(v,x) = L(v_1, \cdots, v_n, x_1, \cdots, x_n)$$

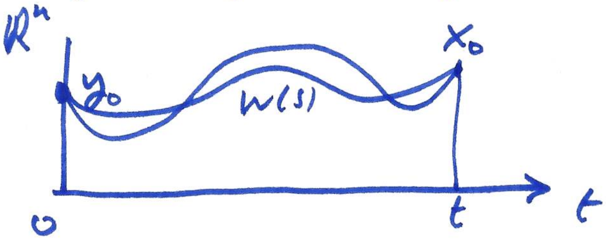
$$\nabla_v L = (L_{v_1}, \cdots, L_{v_n}), \quad \nabla_x L = (L_{x_1}, \cdots, L_{x_n}).$$

Define for any given $x_0, y_0 \in \mathbb{R}^n$, $(x \neq y)$, $t>0$:

$$I[w] = \int_0^t L[\dot{w}(s), w(s)]\, ds. \quad \left(\dot{w} = \frac{dw}{ds}\right)$$

where $w(s) = (w_1(s), w_2(s), \cdots, w_n(s)) \in \mathcal{A}$

$$\mathcal{A} = \{ w \in C^2([0,t]; \mathbb{R}^n) : w(0) = y_0,\ w(t) = x_0 \}$$



call I an action (functional).

Assume $x \in \mathcal{A}$ minimize $I[\cdot]$ over $\mathcal{A}$:

$$I[x] = \min_{w \in \mathcal{A}} I[w]$$

Let $y : [0,t] \to \mathbb{R}^n$ be smooth and $y(0)=0$, $y(t)=0$.

Consider $x + \varepsilon y \in \mathcal{A}$.

$\sigma(\varepsilon) \equiv I[x+\varepsilon y]$ is minimized at $\varepsilon = 0$.

So, $\sigma'(0)=0$.

$$\sigma(\varepsilon) = \int_0^t L(\dot{x}(s)+\varepsilon \dot{y}(s), \cancel{ds}\ x(s)+\varepsilon y(s))\,ds$$

$$\sigma'(\varepsilon) = \int_0^t \Big(\sum_{i=1}^n L_{v_i}(\dot{x}+\varepsilon\dot{y}, x+\varepsilon y)\,\dot{y}_i + \sum_{i=1}^n L_{x_i}(\dot{x}+\varepsilon\dot{y}, x+\varepsilon y)\, y_i \Big)\,ds$$

$$\sigma'(0)=0 \Rightarrow \int_0^t \sum_{i=1}^n \big(L_{v_i}(\dot{x}, x)\,\dot{y}_i + L_{x_i}(\dot{x},x)\,y_i \big)\,ds = 0$$

Integration by parts:

$$\sum_{i=1}^n \int_0^t \Big[\frac{d}{ds}\big(L_{v_i}(\dot{x},x)\big) + L_{x_i}(\dot{x},x) \Big]\, y_i \, ds = 0$$

Hence, $x = x(s)$ $(0 \le s \le t)$ satisfies the <u>Euler-Lagrange eq.</u>

$$-\frac{d}{ds}\big(L_{v_i}(\dot{x},x)\big) + L_{x_i}(\dot{x},x) = 0 \qquad (i=1,2,\cdots,n)$$

<u>Example</u> $L(v,x) = \frac{1}{2} m |v|^2 - \phi(x)$

Euler-Lagrange: $m\ddot{x}(s) = -\nabla\phi(x(s))$

Newton's law of motion.

Now, let us connect Lagrangians to Hamiltonians.

<u>Definition</u> ① For each path $x = x(s)$, $(0 \le s \le t)$, we define $p(s) = \nabla_v L(\dot{x}(s), x(s))$ $(0 \le s \le t)$, and call it the generalized momentum.

② The Hamiltonian H associated with the Lagrangian L is

$$\boxed{H(p,x) = p \cdot v(p,x) - L(v(p,x), x),} \quad (p, x \in \mathbb{R}^n)$$

where $v(p,x)$ solves $p = \nabla_v L(v,x)$

<u>Example</u>

$$L = \frac{1}{2} m |v|^2 - \phi(x)$$

$$p = \nabla_v L(v,x) = mv. \Rightarrow v = \frac{p}{m}.$$

$$H(p,x) = p \cdot \frac{p}{m} - \left(\frac{1}{2} m \cdot \frac{|p|^2}{m^2} - \phi(x)\right)$$

$$= \underbrace{\frac{1}{2m}|p|^2}_{\text{kinetic energy}} + \underbrace{\phi(x)}_{\text{potential energy}}.$$

<u>Thm</u>

$$\dot{x}(s) = \nabla_p H(p(s), x(s))$$
$$\dot{p}(s) = -\nabla_x H(p(s), x(s))$$

<u>Pf</u> Use the Euler-Lagrange equation for $L(\dot{x}, x)$, the definition $H(p,x) = p \cdot v(p,x) - L(v(p,x), x)$, and the definition $p = \nabla_v L(v,x)$. □

Legendre transform

Given $L: \mathbb{R}^n \to \mathbb{R}$. Assume ① it is convex and

② $\lim_{|v|\to\infty} \frac{L(v)}{|v|} = +\infty$.

The Legendre transform of L is $L^*: \mathbb{R}^n \to \overline{\mathbb{R}}$

$$L^*(p) = \sup_{v \in \mathbb{R}^n} (p \cdot v - L(v)) \quad (\forall p \in \mathbb{R}^n)$$

Thm. $(L^*)^* = L$.

Hopf-Lax formula

$$\begin{cases} u_t + H(Du) = 0 & \mathbb{R}^n \times (0,\infty) \\ u = g & \mathbb{R}^n \times \{t=0\} \end{cases}$$

Let $L = H^*$ be the Legendre transform of H. Call it the Lagrangian. Let

$$u(x,t) := \inf \left\{ \int_0^t L(\dot{w}(s))\, ds + g(w(0)) : \begin{matrix} w \in C^1([0,t], \mathbb{R}^n), \\ w(t) = x \end{matrix} \right\}$$

Hopf-Lax formula: If H is convex and $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty$, $g \in C^1$ with $|\nabla g| \le C$ (a const). Then for any $x \in \mathbb{R}^n$, $t > 0$,

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ t\, L\left(\frac{x-y}{t}\right) + g(y) \right\}.$$

Thm (continuing from the formula) If u is differentiable at (x,t) then

$$u_t(x,t) + H(\nabla u(x,t)) = 0.$$