Lecture Notes 4/6/2018, B. Li

Classification of Second-Order Equations

First, let us consider $n=2$; $u=u(x,y)$.
Assume a_{ij}, a_i are all real numbers, and consider

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = 0.$$

We assume a_{11}, a_{12}, a_{22} are not all 0.

Theorem By a linear transformation $(x,y)\longrightarrow(\xi,\eta)$, the equation can be reduced to one of the following three forms for $\tilde{u}(\xi,\eta)=u(x,y)$:

(i) Elliptic, if $a_{12}^2 < a_{11}a_{22}$, and the reduced form is

$$\tilde{u}_{\xi\xi} + \tilde{u}_{\eta\eta} + \text{lower-order terms} = 0;$$

(ii) Hyperbolic, if $a_{12}^2 > a_{11}a_{22}$, and the reduce form is

$$\tilde{u}_{\xi\xi} - \tilde{u}_{\eta\eta} + \text{lower-order terms} = 0;$$

(iii) Parabolic, if $a_{12}^2 = a_{11}a_{22}$, and the reduced form is

$$\tilde{u}_{\xi\xi} + \text{lower-order terms} = 0.$$

The proof of this theorem is done by "completing the square". For instance, suppose $a_{11}=1$ and $a_1=a_2=a_0=0$ (since lower-order terms will not matter). The equation is then

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0.$$

We have

$$u_{xx} + 2a_{12}u_{xy} + a_{12}^2 u_{yy} - a_{12}^2 u_{yy} + a_{22}u_{yy} = 0,$$

i.e., $$(\partial_x + a_{12}\partial_y)^2 u + (a_{22} - a_{12}^2)u_{yy} = 0.$$

Assume $a_{12}^2 < a_{22}$ (the elliptic case; the other cases can be discussed similarly). Let $b = \sqrt{a_{22} - a_{12}^2} > 0$.
The equation becomes

$$(\partial_x + a_{12}\partial_y)^2 u + (b\,\partial_y)^2 u = 0.$$

Let $x = A\xi + B\eta$ $\quad \tilde{u}(\xi, \eta) = u(x, y)$.
$y = C\xi + D\eta$

We need to determine the constants A, B, C, D.
Note that

$$\partial_\xi \tilde{u} = \partial_x u \frac{\partial x}{\partial \xi} + \partial_y u \frac{\partial y}{\partial \xi} = A\partial_x u + C\partial_y u$$

This can be written as

$$\partial_\xi = A\partial_x + C\partial_y.$$

Now, $\partial_\xi^2 \tilde{u} = \partial_\xi(\partial_\xi \tilde{u}) = \partial_\xi(A\partial_x u + C\partial_y u)$

$$= A\,\partial_\xi(\partial_x u) + C\,\partial_\xi(\partial_y u)$$
$$= A(A\partial_x + C\partial_y)\partial_x u + C(A\partial_x + C\partial_y)\partial_y u$$
$$= A^2\partial_x^2 u + 2AC\partial_{xy} u + C^2\partial_y^2 u$$
$$= (A\partial_x + C\partial_y)^2 u$$

i.e., $\partial_\xi^2 = (A\partial_x + C\partial_y)^2$

Similarly, $\partial_\eta^2 = (B\partial_x + D\partial_y)^2$.

Thus, the transformed eq. is simplied, if, by comparing

$$\partial_\xi = \partial_x + a_{12}\partial_y$$
$$\partial_\eta = b\,\partial_y.$$

$A = 1$, $C = a_{12}$, $B = 0$ $\quad D = b$.

Hence with $x = \xi$, $y = a_{12}\xi + b\eta$.
[This is a nonsingular linear transformation!]
we have

$$\partial_\xi^2 \hat{u} + \partial_\eta \hat{u} = 0,$$

as desired.

A second-order equation

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + \text{lower-order terms} = 0$$

is classified therefore as

(i) elliptic, if $a_{12}^2 < a_{11}a_{22}$;

(ii) hyperbolic, if $a_{12}^2 > a_{11}a_{22}$;

(iii) parabolic, if $a_{12}^2 = a_{11}a_{22}$.

(where a_{11}, a_{22}, a_{12} are not all 0.)

Consider now a general $n \geq 2$, and the second-order equation for $u = u(x)$, $x = (x_1, \cdots, x_n)$:

$$(*) \qquad \sum_{i,j=1}^{n} a_{ij}u_{x_ix_j} + \text{lower-order terms} = 0,$$

where $a_{ij} = a_{ji} \in \mathbb{R}$, $i,j = 1, \cdots, n$. $A = [a_{ij}]_{n\times n} \neq 0$. The real, symmetric matrix A can be always diagonalized by a nonsingular linear transformation $x \to \xi$, and the equation can be thus reduced to

$$\sum_{i=1}^{n} \lambda_i \tilde{u}_{\xi_i\xi_i} + \text{lower-order terms} = 0,$$

where $\lambda_1, \cdots, \lambda_n$ are exactly the eigenvalues of A (they are real).

<u>Definition</u> The equation (*) is
(i) elliptic, if all $\lambda_j > 0$ or all $\lambda_j < 0$.
(ii) hyperbolic, if all $\lambda_j \neq 0$ and one of them has the opposite sign from the rest of $(n-1)$ λ_j's;
(iii) parabolic, if exactly one $\lambda_j = 0$ and the rest $(n-1)$ λ_j's have the same sign.

<u>Remark</u> If $a_{ij} = a_{ij}(x)$, then we can define the type of equations at x. (or a region of x).

<u>Example</u> $u = u(x,y)$. $y u_{xx} - 2 u_{xy} + x u_{yy} = 0$.
$a_{12} = -1$, $a_{11} = y$, $a_{22} = x$

$$D = (-1)^2 - yx = 1 - xy.$$ The eq. is:

(i) elliptic in the region $xy > 1$.
(ii) hyperbolic in the region $xy < 1$.
(iii) parabolic on $xy = 1$.