Classification of Second-Order Equations

First, let us consider \( n = 2 \), \( u = u(x, y) \).
Assume \( a_{ij}, a_i \) are all real numbers, and consider
\[ a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + a_1 u_x + a_2 u_y + q_0 u = 0. \]
We assume \( a_{11}, a_{12}, a_{22} \) are not all 0.

**Theorem** By a linear transformation \((x, y) \rightarrow (\xi, \eta)\), the equation can be reduced to one of the following three forms for \( u(\xi, \eta) = u(x, y) \):

(i) **Elliptic**, if \( a_{11} \underline{a_{12}} a_{22} \), and the reduced form is
\[ \tilde{u}_{\xi\xi} + \tilde{u}_{\eta\eta} + \text{lower-order terms} = 0; \]
(ii) **Hyperbolic**, if \( a_{11} \underline{a_{12}} a_{22} \), and the reduced form is
\[ \tilde{u}_{\xi\xi} - \tilde{u}_{\eta\eta} + \text{lower-order terms} = 0; \]
(iii) **Parabolic**, if \( a_{12} = a_{11} a_{22} \), and the reduced form is
\[ \tilde{u}_{\xi\xi} + \text{lower-order terms} = 0. \]

The proof of this theorem is done by "completing the square". For instance, suppose \( a_{11} = 1 \) and \( a_1 = a_2 = a_0 = 0 \) (since lower-order terms will not matter). The equation is then
\[ u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} = 0. \]

We have
\[ u_{xx} + 2a_{12} u_{xy} + a_{11} a_{22} u_{yy} - a_{12}^2 u_{yy} + a_{22} u_{yy} = 0, \]
\[ \text{i.e.,} \quad (\partial_x^2 + a_{12}^2 \partial_y^2) u_{\xi} + (a_{11} - a_{12}^2) u_{\eta\eta} = 0. \]
Assume $a_{12}^2 < a_{22}$ (the elliptic case; the other cases can be discussed similarly). Let $b = \sqrt{a_{22} - a_{12}^2} > 0$.

The equation becomes

$$(2x + a_{12} \partial y)^2 u + (b \partial y)^2 u = 0.$$ 

Let $x = A \xi + B \eta$ and $u(\xi, \eta) = u(x, y)$.

We need to determine the constants $A, B, C, D$.

Note that

$$\partial_\zeta \tilde{u} = \partial_x u \frac{\partial x}{\partial \zeta} + \partial_y u \frac{\partial y}{\partial \zeta} = A \partial_x u + C \partial_y u$$

This can be written as

$$\partial_\zeta = A \partial_x + C \partial_y.$$ 

Now, $\partial_\zeta^2 \tilde{u} = \partial_\zeta (\partial_\zeta \tilde{u}) = \partial_\zeta (A \partial_x u + C \partial_y u)$

$$= A \partial_\zeta (\partial_x u) + C \partial_\zeta (\partial_y u)$$

$$= A (A \partial_x + C \partial_y) \partial_x u + C (A \partial_x + C \partial_y) \partial_y u$$

$$= A^2 \partial_x^2 u + 2AC \partial_x \partial_y u + C^2 \partial_y^2 u$$

$$= (A \partial_x + C \partial_y)^2 u$$

i.e., $\partial_\zeta^2 = (A \partial_x + C \partial_y)^2$.

Similarly, $\partial_\eta^2 = (B \partial_x + D \partial_y)^2$.

Thus, the transformed eq. is simplified, if, by comparing $\partial_\zeta = A \partial_x + a_{12} \partial_y$

$$\partial_\eta = b \partial_y.$$ 

$A = 1$, $C = a_{12}$, $B = 0$, $D = b$.

Hence, with $x = \xi$, $y = a_{12} \xi + b \eta$,

[This is a nonsingular linear transformation!]

we have

$$\partial_\zeta^2 \tilde{u} + \partial_\eta \tilde{u} = 0,$$

as desired.
A second-order equation
\[ a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + \text{lower-order terms} = 0 \]
is classified therefore as

(i) **elliptic**, if \( a_{12}^2 < a_{11} a_{22} \);
(ii) **hyperbolic**, if \( a_{12}^2 > a_{11} a_{22} \);
(iii) **parabolic**, if \( a_{12}^2 = a_{11} a_{22} \).

(where \( a_{11}, a_{12}, a_{22} \) are not all 0.)

Consider now a general \( n \geq 2 \), and the second-order equation for \( u = u(x) \),
\[ x = (x_1, \ldots, x_n) \):

\( \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \text{lower-order terms} = 0 \)

where \( a_{ij} = a_{ji} \in \mathbb{R}, i,j = 1, \ldots, n \). \( A = [a_{ij}]_{n \times n} \neq 0 \).

The real, symmetric matrix \( A \) can be always diagonalized by a nonsingular linear transformation \( x \to z \), and the equation can be thus reduced to
\[ \sum_{i=1}^{n} \lambda_i u_{z_i z_i} + \text{lower-order terms} = 0, \]

where \( \lambda_1, \ldots, \lambda_n \) are exactly the eigenvalues of \( A \) (they are real).
Definition: The equation (1) is

(i) elliptic, if all \( \lambda_j > 0 \) or all \( \lambda_j < 0 \).

(ii) hyperbolic, if all \( \lambda_j \neq 0 \) and one of them has the opposite sign from the rest of \( (n-1) \lambda \)'s.

(iii) parabolic, if exactly one \( \lambda_j = 0 \) and the rest \( (n-1) \lambda \)'s have the same sign.

Remark: If \( a_{ij} = a_{ij}(x) \), then we can define the type of equation at \( x \) (or a region of \( x \)).

Example:
\[ u = u(xy) \]
\[ y u_{xx} - 2 u_{xy} + x u_{yy} = 0. \]
\[ a_{11} = -1, \quad a_{12} = y, \quad a_{22} = x. \]
\[ D = (-1)^2 - y x = 1 - xy. \] The eq. is:

(i) elliptic in the region \( xy > 1 \).

(ii) hyperbolic in the region \( xy < 1 \).

(iii) parabolic on \( xy = 1 \).

\[ \text{elliptic} \]
\[ \text{hyperbolic} \]
\[ \text{elliptic} \]