Math 210C: Mathematical Methods in Physical Sciences and Engineering
Spring quarter, 2018

Homework Assignment 3

Due Wednesday, April 25, 2018

1. For each integer $n \geq 1$, define $u_n, v_n : \mathbb{R}^2 \to \mathbb{R}$ in the polar coordinates by $u_n(r, \theta) = r^n \cos(n\theta)$ and $v_n(r, \theta) = r^n \sin(n\theta)$. Verify that both u_n and v_n are harmonic functions in $\mathbb{R}^2$.

2. Let $u \in C^2(\mathbb{R}^2)$. Let $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ be smooth functions that define a bijective map from $\mathbb{R}^2$ to $\mathbb{R}^2$ with a smooth inverse $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$. Define $v(\xi, \eta) = u(x, y)$ with $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ for for any $(\xi, \eta) \in \mathbb{R}^2$.
   (1) Verify that
   $$\Delta u(x, y) = v_{\xi\xi}|\nabla_{(x,y)}\xi|^2 + v_{\eta\eta}|\nabla_{(x,y)}\eta|^2 + 2v_{\xi\eta}\nabla_{(x,y)}\xi \cdot \nabla_{(x,y)}\eta + v_\xi \Delta_{(x,y)}\xi + v_\eta \Delta_{(x,y)}\eta.$$
   (2) Use Part (1) to show that the Laplacian in the polar coordinates (r, θ) is given by
   $$\Delta_{(x,y)}u(r, \theta) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

3. Let $n \geq 2$ be an integer and $r = |x| = \sqrt{x_1^2 + \dots x_n^2}$ with $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Let $u \in C^2(\mathbb{R})$.
   (1) Express $\Delta u = \sum_{j=1}^n \partial^2 u / \partial x_j^2$ in terms of $u'(r)$ and $u''(r)$.
   (2) Solve $\Delta u = 0$ by finding the general solution to the ordinary differential equation for $u = u(r)$.

4. Solve Laplace's equation $\Delta u = 0$ on the disk $r < 1$ with the boundary condition $u = 1 + 3\sin\theta - 4\cos 5\theta$ at $r = 1$.

5. (1) Show that $K(x) = -|x|/2$ satisfies $-K'' = \delta$ in $\mathbb{R}$, where δ is the one-dimensional Dirac delta function at 0. This means that
   $$\int_{-\infty}^{\infty} K'(x)\phi'(x)\,dx = \phi(0)$$
   for any continuously differentiable and compactly supported function $\phi = \phi(x)$.
   (2) Construct the Green's function $G = G(x, y)$ on a finite interval (a, b) as
   $$G(x, y) = K(x - y) + g^x(y) \quad \forall y \in (a, b)$$
   where g^x for each $x \in (a, b)$ is a harmonic function such that $G(x, a) = G(x, b) = 0$.

6. (1) Let $v = v(x, y)$ be a harmonic function. Prove that $u(x, y) = v(x^2 - y^2, 2xy)$ is also a harmonic function.
   (2) Prove that the mapping $(x, y) \mapsto (x^2 - y^2, 2xy)$ maps the first quadrant $x > 0$ and $y > 0$ to the upper half plan $y > 0$.
   (3) Use the mapping in Part (2) and the Green's function for the upper half plane to construct the Green's function for the first quadrant.
   (4) Solve Laplace's equation $\Delta u = 0$ in the region $x > 0$ and $y > 0$ with the boundary conditions $u(0, y) = g(y)$ for $y > 0$ and $u(x, 0) = h(x)$ for $x > 0$, where g and h are given continuous functions.