1. For each integer \( n \geq 1 \), define \( u_n, v_n : \mathbb{R}^2 \to \mathbb{R} \) in the polar coordinates by \( u_n(r, \theta) = r^n \cos(n \theta) \) and \( v_n(r, \theta) = r^n \sin(n \theta) \). Verify that both \( u_n \) and \( v_n \) are harmonic functions in \( \mathbb{R}^2 \).

2. Let \( u \in C^2(\mathbb{R}^2) \). Let \( \xi = \xi(x, y) \) and \( \eta = \eta(x, y) \) be smooth functions that define a bijective map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) with a smooth inverse \( x = x(\xi, \eta) \) and \( y = y(\xi, \eta) \). Define \( v(\xi, \eta) = u(x, y) \) with \( x = x(\xi, \eta) \) and \( y = y(\xi, \eta) \) for any \((\xi, \eta) \in \mathbb{R}^2\).

   (1) Verify that \[
   \Delta u(x, y) = v_\xi \left[ \nabla_{(x,y)} \xi \right]^2 + v_\eta \left[ \nabla_{(x,y)} \eta \right]^2 + 2v_\xi \eta \nabla_{(x,y)} \xi \cdot \nabla_{(x,y)} \eta + v_\xi \Delta_{(x,y)} \xi + v_\eta \Delta_{(x,y)} \eta.
   \]

   (2) Use Part (1) to show that the Laplacian in the polar coordinates \((r, \theta)\) is given by
   \[
   \Delta_{(r, \theta)} u(r, \theta) = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta}.
   \]

3. Let \( n \geq 2 \) be an integer and \( r = |x| = \sqrt{x_1^2 + \ldots + x_n^2} \) with \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). Let \( u \in C^2(\mathbb{R}) \).

   (1) Express \( \Delta u = \sum_{j=1}^n \partial^2 u / \partial x_j^2 \) in terms of \( u'(r) \) and \( u''(r) \).

   (2) Solve \( \Delta u = 0 \) by finding the general solution to the ordinary differential equation for \( u = u(r) \).

4. Solve Laplace’s equation \( \Delta u = 0 \) on the disk \( r < 1 \) with the boundary condition \( u = 1 + 3 \sin \theta - 4 \cos 5\theta \) at \( r = 1 \).

5. (1) Show that \( K(x) = -|x|/2 \) satisfies \(-K'' = \delta \) in \( \mathbb{R} \), where \( \delta \) is the one-dimensional Dirac delta function at 0. This means that
   \[
   \int_{-\infty}^{\infty} K'(x) \phi'(x) \, dx = \phi(0)
   \]
   for any continuously differentiable and compactly supported function \( \phi = \phi(x) \).

   (2) Construct the Green’s function \( G = G(x, y) \) on a finite interval \((a, b)\) as
   \[
   G(x, y) = K(x - y) + g^x(y) \quad \forall y \in (a, b)
   \]
   where \( g^x \) for each \( x \in (a, b) \) is a harmonic function that vanishes at \( a \) and \( b \).

6. (1) Let \( v = v(x, y) \) be a harmonic function. Prove that \( u(x, y) = v(x^2 - y^2, 2xy) \) is also a harmonic function.

   (2) Prove that the mapping \((x, y) \mapsto (x^2 - y^2, 2xy)\) maps the first quadrant \( x > 0 \) and \( y > 0 \) to the upper half plane \( y > 0 \).

   (3) Use the mapping in Part (2) and the Green’s function for the upper half plane to construct the Green’s function for the first quadrant.

   (4) Solve Laplace’s equation \( \Delta u = 0 \) in the region \( x > 0 \) and \( y > 0 \) with the boundary conditions \( u(0, y) = g(y) \) for \( y > 0 \) and \( u(x, 0) = h(x) \) for \( x > 0 \), where \( g \) and \( h \) are given continuous functions.