1. Let $\alpha > 0$ and define $K(r) = (1/r)e^{-\alpha r} (r > 0)$. Verify that

$$-\Delta K(|x|) + \alpha^2 K(|x|) = 0 \quad \forall x \in \mathbb{R}^3, \ x \neq 0.$$ 

Optional: Prove that $K(|x|)$ is the fundamental solution to $-\Delta + \alpha$ in $\mathbb{R}^3$, i.e.,

$$-\Delta K + \alpha^2 K = \delta \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad K(\infty) = 0.$$

2. Let $\alpha > 0$. Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be two given functions. Prove the uniqueness of solution to the Robin boundary-value problem of Poisson’s equation in a bounded and smooth domain $\Omega$:

$$\begin{cases}
- \Delta u = f & \text{in } \Omega, \\
\partial_n u + \alpha u = g & \text{on } \partial\Omega.
\end{cases}$$

3. Let $\kappa$ be a positive number. Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be two given functions. Prove the uniqueness of solution to the boundary-value problem:

$$\begin{cases}
- \Delta u + \kappa^2 u = f & \text{in } \Omega, \\
u = g & \text{on } \partial\Omega,
\end{cases}$$

or

$$\begin{cases}
- \Delta u + \kappa^2 u = f & \text{in } \Omega, \\
\partial_n u = g & \text{on } \partial\Omega.
\end{cases}$$

4. Let $\Omega$ be a bounded and smooth domain in $\mathbb{R}^d$ for some $d \geq 2$. Calculate the Euler–Lagrange equation for the functional

$$E[u] = \int_{\Omega} \frac{1}{2} [|\Delta u|^2 - \ln (1 + |\nabla u|^2)] \, dx.$$ 

5. Is the statement of the Mean-Value Theorem for a harmonic function still true if the sphere is replaced by a cube or a disk is replaced by a square?