Math 210C: Mathematical Methods in Physical Sciences and Engineering
Spring quarter, 2018

Homework Assignment 4

Due Friday, May 4, 2018

1. Let $\alpha > 0$ and define $K(r) = (1/4\pi r)e^{-\alpha r}$ $(r > 0)$. Verify that

$$-\Delta K(|x|) + \alpha^2 K(|x|) = 0 \qquad \forall x \in \mathbb{R}^3,\ x \neq 0.$$

Optional: Prove that $K(|x|)$ is the fundamental solution to $-\Delta + \alpha$ in $\mathbb{R}^3$, i.e.,

$$-\Delta K + \alpha^2 K = \delta \quad \text{in } \mathbb{R}^3 \qquad \text{and} \qquad K(\infty) = 0.$$

2. Let $\alpha > 0$. Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be two given functions. Prove the uniqueness of solution to the Robin boundary-value problem of Poisson's equation in a bounded and smooth domain Ω :

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_n u + \alpha u = g & \text{on } \partial\Omega. \end{cases}$$

3. Let κ be a positive number. Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be two given functions. Prove the uniqueness of solution to the boundary-value problem:

$$\begin{cases} -\Delta u + \kappa^2 u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

or

$$\begin{cases} -\Delta u + \kappa^2 u = f & \text{in } \Omega, \\ \partial_n u = g & \text{on } \partial\Omega. \end{cases}$$

4. Let Ω be a bounded and smooth domain in $\mathbb{R}^d$ for some $d \geq 2$. Calculate the Euler–Lagrange equation for the functional

$$E[u] = \int_\Omega \frac{1}{2}\left[|\Delta u|^2 - \ln\left(1 + |\nabla u|^2\right)\right] dx.$$

5. Is the statement of the Mean-Value Theorem for a harmonic function still true if the sphere is replaced by a cube or a disk is replaced by a square?

6. (Optional) Let $f \in C(\mathbb{R} \times [0, \infty))$ and $g \in C(\mathbb{R})$. Show that the boundary-value problem $\Delta u = f$ in $\mathbb{R} \times (0, \infty)$ and $u(\cdot, 0) = g$ on $\mathbb{R}$ can have at most one solution $u \in C(\mathbb{R} \times [0, \infty)) \cap C^2(\mathbb{R} \times (0, \infty))$. that satisfies $u(x, y) \to 0$ as $y \geq 0$ and $x^2 + y^2 \to \infty$.