Math 210C: Mathematical Methods in Physical Sciences and Engineering
Spring quarter, 2018

Homework Assignment 5
Due Monday, May 14, 2018

1. Let $f \in C([0,1])$ be given. Use the method of separation of variables to find the series expansion of the solution $u = u(x,t)$ to the following initial-boundary-value problem:

$$\begin{cases} u_t = u_{xx} - u & \text{for } 0 < x < 1, t > 0, \\ u(0,t) = u(1,t) = 0 & \text{for } t > 0, \\ u(x,0) = f(x) & \text{for } 0 < x < 1. \end{cases}$$

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain. Let $u \in C^2(\overline{\Omega})$ be a nonzero (real-valued) function and let $\lambda \in \mathbb{R}$. Suppose $-\Delta u = \lambda u$ in Ω and $u = 0$ on $\partial\Omega$. Show that

$$\lambda = \left(\int_\Omega |\nabla u|^2 dx \right) \left(\int_\Omega u^2 dx \right)^{-1} > 0.$$

3. Find the Fourier transform for each of the following functions:
   (1) $f(x) = (1/2)\chi_{[-1,1]}$, where χ_A denotes the characteristic function of A (i.e., $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$);
   (2) (Optional) $f(x) = 1/(1+x^2)$ $(x \in \mathbb{R})$.

4. (1) The Gaussian kernel (or heat kernel) in one space dimension is defined to be

$$K(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \quad (x \in \mathbb{R}, t > 0),$$

   where $D > 0$ is a constant. Verify that $K_t = DK_{xx}$ for all $x \in \mathbb{R}$ and $t > 0$.
   (2) Define now $K_n(x,t) = \prod_{i=1}^n K(x_i, t)$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t > 0$. Verify that $u_t = D\Delta u$ for all $x \in \mathbb{R}^n$ and $t > 0$.

5. Use the fundamental solution to the heat equation to find a formula of solution to the following initial-boundary-value problem of the heat equation on half-line:

$$\begin{cases} u_t = Du_{xx} & \text{for } x > 0, t > 0, \\ u(0,t) = 0 & \text{for } t > 0, \\ u(x,0) = \phi(x) & \text{for } x > 0, \end{cases}$$

   where $D > 0$ is a constant and ϕ is a compactly supported continuous function on $[0, \infty)$.

6. Let Ω be a bounded and smooth domain in $\mathbb{R}^d$ and denote by n the unit exterior normal to the boundary $\partial\Omega$. Suppose $u = u(x,t)$ $(x \in \overline{\Omega}, t \geq 0)$ is a smooth and bounded function. Suppose also that

$$\begin{cases} u_t = D\Delta u & \text{for } x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial n} = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = \phi(x) & \text{for } x \in \Omega, \end{cases}$$

   for some $\phi \in C(\overline{\Omega})$. Show that

$$\int_\Omega u(x,t)\, dx = \int_\Omega \phi(x)\, dx \qquad \forall t > 0.$$