Math 210C: Mathematical Methods in Physical Sciences and Engineering
Spring quarter, 2018

Homework Assignment 6
Due Wednesday, May 30, 2018

1. The Fourier transform for any $u \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx \qquad \forall \xi \in \mathbb{R}^n.$$

Prove the following:

(1) For any $h \in \mathbb{R}^n$, $\lambda > 0$, and $u \in L^1(\mathbb{R}^n)$, $\widehat{\tau_h u}(\xi) = e^{-ih\cdot\xi}\hat{u}(\xi)$ and $\widehat{\delta_\lambda u}(\xi) = \lambda^n \hat{u}(\lambda\xi)$ $(\xi \in \mathbb{R}^n)$, where $\tau_h u(x) = u(x-h)$ and $\delta_\lambda u(x) = u(x/\lambda)$.
(2) For any $u \in C_c^2(\mathbb{R}^n)$, $\widehat{\Delta u}(\xi) = -|\xi|^2 \hat{u}(\xi)$ $(\xi \in \mathbb{R}^n)$.
(3) If $u,\, v \in C_c(\mathbb{R}^n)$, then $\widehat{u * v} = (2\pi)^{n/2}\hat{u}\hat{v}$.

2. Let $D > 0$ and $\alpha > 0$ be two given constants, and consider the diffusion equation

$$u_t = Du_{xx} + \alpha u \qquad (x \in R,\, t > 0).$$

Let $k > 0$ and define $u_k(x,t) = e^{\omega t}\sin(kx)$ $(x \in \mathbb{R},\, t > 0)$, where ω is a constant to be determined.

(1) Find the formula for $\omega = \omega(k, D, \alpha)$ so that $u_k(x,t)$ solves the above diffusion equation.
(2) With that $\omega = \omega(k, D, \alpha)$, find all $k > 0$ such that $u_k(x,t)$ are bounded as $t \to \infty$.

3. Let $f \in C(\mathbb{R}^n)$ be bounded. Let $u = u(x,t)$ be the solution to the heat equation $u_t = \Delta u$ $(x \in \mathbb{R}^n,\, t > 0)$ with the initial condition $u(x,0) = f(x)$ $(x \in \mathbb{R}^n)$, given by

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t) f(y)\, dy \qquad \forall t > 0,$$

where K is the heat kernel.

(1) Show that $|u(x,t)| \le \sup_{y\in\mathbb{R}^n} |f(y)|$ for all $x \in \mathbb{R}^n$ and $t \ge 0$.
(2) Assume in addition $f \in L^1(\mathbb{R}^n)$. Show that $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in \mathbb{R}^n$.

4. Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, $D > 0$ and $T > 0$ constants, $f \in C(\overline{\Omega} \times [0,T])$, $g \in C(\partial\Omega \times [0,T])$, and $\phi \in C(\overline{\Omega})$. Use the energy method to prove the uniqueness of solution to the initial-boundary-value problem: $u_t - D\Delta u = f$ in $\Omega \times (0,T]$, $u = g$ on $\partial\Omega \times (0,T]$, and $u = \phi$ on $\Omega \times \{0\}$.

5. (The Markov property of solutions to diffusion equations.) Let $u = u(x,t)$ solve the diffusion equation $u_t = D\Delta u$ in $\Omega \times (0,\infty)$, with the zero Dirichlet boundary condition $u(x,t) = 0$ $(x \in \partial\Omega, t > 0)$, where $D > 0$ is the diffusion constant and Ω is a bounded and smooth domain in $\mathbb{R}^n$. Let $t_1 > 0$ and let $u_1 = u_1(x,t)$ solve the diffusion equation $u_{1t} = D\Delta u_1$ in $\Omega \times (0,\infty)$, with the zero Dirichlet boundary condition $u_1(x,t) = 0$ $(x \in \partial\Omega,\, t > 0)$ and the initial condition $u_1(x,0) = u(x,t_1)$ $(x \in \Omega)$. Prove that $u(x, t_1 + t_2) = u_1(x, t_2)$ for any $x \in \Omega$ and any $t_2 > 0$.

6. Let $D > 0$, $\kappa > 0$, $Y(x,t) = e^{-\kappa t}K(x,t)$, and $K(x,t) = (4\pi Dt)^{-n/2} e^{-\frac{|x|^2}{4Dt}}$ $(x \in \mathbb{R}^n,\, t > 0)$.

(1) Verify that $Y_t - D\Delta Y + \kappa Y = 0$ in $\mathbb{R}^n \times (0,\infty)$.
(2) Let $f \in C(\mathbb{R}^n)$ be bounded. Use the kernel $Y(x,t)$ to find a formula of the solution to the initial-value problem

$$\begin{cases} u_t - \Delta u + \kappa u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = f & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$