Math 210C: Mathematical Methods in Physical Sciences and Engineering  
Spring quarter, 2018  
Homework Assignment 6  
Due Wednesday, May 30, 2018

1. The Fourier transform for any $u \in L^1(\mathbb{R}^n)$ is defined by
   $$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-i x \cdot \xi} dx \quad \forall \xi \in \mathbb{R}^n.$$ 

   Prove the following:
   
   (1) For any $h \in \mathbb{R}^n$, $\lambda > 0$, and $u \in L^1(\mathbb{R}^n)$, $\hat{\tau_h u}(\xi) = e^{-i h \cdot \xi} \hat{u}(\xi)$ and $\hat{\delta_{\lambda} u}(\xi) = \lambda^n \hat{u}(\lambda \xi)$ ($\xi \in \mathbb{R}^n$), where $\tau_h u(x) = u(x - h)$ and $\delta_{\lambda} u(x) = u(x/\lambda)$.
   
   (2) For any $u \in C^2_c(\mathbb{R}^n)$, $\hat{\Delta u}(\xi) = -|\xi|^2 \hat{u}(\xi)$ ($\xi \in \mathbb{R}^n$).
   
   (3) If $u, v \in C^2_c(\mathbb{R}^n)$, then $\hat{u} \ast \hat{v} = (2\pi)^{n/2} \hat{u} \hat{v}$.

2. Let $D > 0$ and $\alpha > 0$ be two given constants, and consider the diffusion equation
   $$u_t = Du_{xx} + \alpha u \quad (x \in R, t > 0).$$

   Let $k > 0$ and define $u_k(x, t) = e^{-\omega t} \sin(kx)$ ($x \in \mathbb{R}$, $t > 0$), where $\omega$ is a constant to be determined.

   (1) Find the formula for $\omega = \omega(k, D, \alpha)$ so that $u_k(x, t)$ solves the above diffusion equation.
   
   (2) With that $\omega = \omega(k, D, \alpha)$, find all $k > 0$ such that $u_k(x, t)$ are bounded as $t \to \infty$.

3. Let $u = u(x, t)$ solve the heat equation $u_t = \Delta u \ (x \in \mathbb{R}^n, t > 0)$ with the initial condition $u(x, 0) = f(x)$ ($x \in \mathbb{R}^n$).

   (1) Let $f \in C(\mathbb{R}^n)$ be bounded. Show that $|u(x, t)| \leq \sup_{y \in \mathbb{R}^n} |f(y)|$ for all $x \in \mathbb{R}^n$ and $t \geq 0$.
   
   (2) (Optional) Assume in addition $f \in L^1(\mathbb{R}^n)$. Show that $\lim_{t \to \infty} u(x, t) = 0$ for all $x \in \mathbb{R}^n$.

4. Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, $D > 0$ and $T > 0$ constants, $f \in C(\bar{\Omega} \times [0, T])$, $g \in C(\partial \Omega \times [0, T])$, and $\phi \in C(\bar{\Omega})$. Use the energy method to prove the uniqueness of solution to the initial-boundary-value problem: $u_t - D\Delta = f$ in $\Omega \times (0, T)$, $u = g$ on $\partial \Omega \times (0, T)$, and $u = \phi$ on $\Omega \times \{0\}$.

5. (The Markov property of solutions to diffusion equations.) Let $u = u(x, t)$ solve the diffusion equation $u_t = D\Delta u$ in $\Omega \times (0, \infty)$, with the zero Dirichlet boundary condition $u(x, t) = 0$ ($x \in \partial \Omega, t > 0$), where $D > 0$ is the diffusion constant and $\Omega$ is a bounded and smooth domain in $\mathbb{R}^n$. Let $t_1 > 0$ and let $u_1 = u_1(x, t)$ solve the diffusion equation $u_{1t} = D\Delta u_1$ in $\Omega \times (0, \infty)$, with the zero Dirichlet boundary condition $u_1(x, t) = 0$ ($x \in \partial \Omega, t > 0$) and the initial condition $u_1(x, 0) = u(x, t_1)$ ($x \in \Omega$). Prove that $u(x, t_1 + t_2) = u_1(x, t_2)$ for any $x \in \Omega$ and any $t_2 > 0$.

6. Let $D > 0$, $\kappa > 0$, $Y(x, t) = e^{-\kappa t}K(x, t)$, and $K(x, t) = (4\pi Dt)^{-n/2}e^{-|x|^2/4Dt}$ ($x \in \mathbb{R}^n, t > 0$).

   (1) Verify that $Y_t - D\Delta Y + \kappa Y = 0$ in $\mathbb{R}^n \times (0, \infty)$.

   (2) Let $f \in C(\mathbb{R}^n)$ be bounded. Use the kernel $Y(x, t)$ to find a formula of the solution to the initial-value problem
   $$\begin{cases} 
   u_t - \Delta u + \kappa u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
   u = f & \text{on } \mathbb{R}^n \times \{0\}.
   \end{cases}$$