Convergence of Random Variables, Laws of Large Numbers, and Central Limit Theorem

Convergence of Random Variables

If $a=(a_1,\cdots,a_d)\in\mathbb{R}^d$, then we denote
$$|a|=\|a\|=\sqrt{\sum_{i=1}^{d} a_i^2}\,.$$

Definition Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables in $\mathbb{R}^d$ (i.e., each $X_n:\Omega\to\mathbb{R}^d$ is a random variable). Let $X:\Omega\to\mathbb{R}^d$ be a random variable.

(1) The sequence $\{X_n\}_{n=1}^{\infty}$ converges to X pointwise, if
$$\lim_{n\to\infty} X_n(\omega)=X(\omega)\qquad\forall\omega\in\Omega$$

(2) The sequence $\{X_n\}_{n=1}^{\infty}$ converges to X almost surely (a.s), or almost everywhere (a.e.), denoted $X_n\xrightarrow{a.s}X$, if $\exists A_0\in\mathcal{A}$ s.t. $\mathbb{P}(A_0)=0$ and
$$\lim_{n\to\infty} X_n(\omega)=X(\omega)\qquad\forall\omega\in\Omega\setminus A_0.$$

(3) The sequence $\{X_n\}_{n=1}^{\infty}$ converges to X in probability, denoted $X_n\xrightarrow{\mathbb{P}}X$, if for any $\varepsilon>0$,
$$\lim_{n\to\infty}\mathbb{P}(|X_n-X|>\varepsilon)=0.$$

(4) The sequence $\{X_n\}_{n=1}^{\infty}$ converges to X <u>in mean</u>, or <u>in L^1</u>, denoted $X_n \xrightarrow{L^1} X$, if

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|) = 0.$$

(5) The sequence $\{X_n\}_{n=1}^{\infty}$ converges to X <u>in L^r $(r \geq 1)$</u>, denoted $X_n \xrightarrow{L^r} X$, if

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0.$$

(6) The sequence $\{X_n\}_{n=1}^{\infty}$ converges to X <u>in distribution</u>, denoted $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for any $x \in \mathbb{R}^d$ that is a continuity point of $F : \mathbb{R}^d \to [0,1]$, where $F_n : \mathbb{R}^d \to \mathbb{R}$ and $F : \mathbb{R}^d \to [0,1]$ are the distribution functions of X_n and X, respectively.

Relations between different convergence concepts.

$$\begin{array}{l} X_n \xrightarrow{a.s.} X \;\; \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} \\ \\ X_n \xrightarrow{L^r} X \;\; \begin{array}{c}\Rightarrow \\ \not\Leftarrow\end{array} \end{array} \quad X_n \xrightarrow{p} X \Longrightarrow X_n \xrightarrow{d} X$$

If $X_n \xrightarrow{p} X$ then $\exists$ subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $X_{n_k} \xrightarrow{a.s.} X$.

Definition Let $A_k \in \mathcal{A}$ $(k=1,2,\cdots)$.

$$\limsup_{n\to\infty} A_n = \{\omega\in\Omega : \omega\in A_n \text{ for infinitely many } n\}$$

$$\liminf_{n\to\infty} A_n = \{\omega\in\Omega : \omega\in A_n \text{ for all but finitely many } n\}.$$

Clearly

$$\limsup_{n\to\infty} A_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n.$$

$$\liminf_{n\to\infty} A_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n.$$

Moreover, $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.

Definition If $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$, then

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n.$$

Theorem (Borel–Cantelli Lemma) Let $A_k \in \mathcal{A}$ $(k=1, 2,\cdots)$.

(1) If $\sum_{k=1}^{\infty}\mathbb{P}(A_k) < \infty$ then

$$\mathbb{P}\left(\limsup_{k\to\infty} A_k\right) = 0.$$

(2) Assume $A_1, A_2, \cdots$ are independent, then

$$\sum_{k=1}^{\infty}\mathbb{P}(A_k) < \infty \iff \mathbb{P}\left(\limsup_{k\to\infty} A_k\right) = 0,$$

$$\sum_{k=1}^{\infty}\mathbb{P}(A_k) = \infty \iff \mathbb{P}\left(\limsup_{k\to\infty} A_k\right) = 1.$$

Theorem (Lebesgue Monotone Convergence Theorem)
Let $X, X_1, X_2, \ldots$ be all nonnegative, real random variables such that

(1) $X_n \le X_{n+1}$ a.s. $n = 1, 2, \ldots$

(2) $\lim\limits_{n\to\infty} X_n = X$ a.s.

Then $\lim\limits_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X)$.

Theorem (Fatou's Lemma) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of nonnegative random variables. Then $\liminf\limits_{n\to\infty} \mathbb{E}(X_n) \ge \mathbb{E}\left(\liminf\limits_{n\to\infty} X_n\right)$.

Theorem (Lebesgue Dominated Convergence Theorem)
Let $X_n : \Omega \to \mathbb{R}^d$ $(n = 1, 2, \ldots)$ and $X : \Omega \to \mathbb{R}^d$ be all random variables. Suppose:

(1) There exists a random variable $Y : \Omega \to [0, \infty]$ such that $|X_n| \le Y$ a.s., $n = 1, 2, \ldots$, and $\mathbb{E}(Y) < \infty$;

(2) $\lim\limits_{n\to\infty} X_n = X$ a.s. or in probability.

Then $\lim\limits_{n\to\infty} \mathbb{E}(X_n) = \mathbb{E}(X)$.

Laws of Large Numbers and Central Limit Theorems

Theorem (A Weak Law of Large Numbers). Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables in $\mathbb{R}^d$ with a finite mean $\mu = \mathbb{E}(X_n)$ ($n=1, 2, \ldots$). Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_n = \mu \quad \text{in probability.}$$

Theorem (A Strong Law of Large Numbers) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables in $\mathbb{R}^d$ with a finite mean $\mu = \mathbb{E}(X_n)$ ($n=1,2,\ldots$). Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = \mu, \quad \text{a.s.}$$

Theorem (The Central Limit Theorem). Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed, real random variables with finite expectation $\mathbb{E}(X_n)$ ($n=1,2,\ldots$) and finite variance $\sigma^2 = \mathrm{Var}(X_n) > 0$ ($n=1,2,\ldots$). Then

$$\lim_{n\to\infty} \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} = Z \quad \text{in distribution,}$$

for some real random variable Z that is $N(0,1)$ (the standard Gaussian) distributed, where $\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$.

Note that the probability density function for a standard Gaussian random variable is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (x \in \mathbb{R}).$$

The conclusion of the Central Limit Theorem is

$$\lim_{n\to\infty} \mathbb{P}\left(a \le \frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu) \le b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2} dx$$

for any $a, b \in \mathbb{R}$ with $a < b$.

Equivalently,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu) < x\right) = \Phi(x),$$

for any $x \in \mathbb{R}$, where

$$\Phi(x) = \int_{-\infty}^{x} \phi(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2} dy$$

is the cumulative distribution function of a standard Gaussian random variable.