Lecture 15. Friday. 4/29/2022

Today: Finish proving some results used in the proof of existence of an optimal plan.

Set up X, Y: Polish. $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$

$c : X \times Y \to [0, \infty]$: measurable.

$$\mathcal{A}(\mu, \nu) \triangleq \{ \gamma \in \mathcal{P}(X \times Y) : \pi^X_\# \gamma = \mu, \ \pi^Y_\# \gamma = \nu \}$$

$$E_K[\gamma] = \int_{X \times Y} c(x, y) \, d\gamma(x, y), \quad \gamma \in \mathcal{A}(\mu, \nu).$$

Theorem Assume $c : X \times Y \to [0, \infty]$ is lower semi-continuous. Then $\exists \ \hat{\gamma} \in \mathcal{A}(\mu, \nu)$ s.t. $E_K[\hat{\gamma}] = \min_{\gamma \in \mathcal{A}(\mu,\nu)} E_K[\gamma]$.

Proof Let $\gamma_k \in \mathcal{A}(\mu, \nu)$: $E_K(\gamma_k) \to \alpha := \inf_{\gamma \in \mathcal{A}(\mu, \nu)} E_K[\gamma] \in [0, \infty)$.

- $\exists$ subseq. $\gamma_{k_j} \to \hat{\gamma} \in \mathcal{A}(\mu, \nu)$ narrowly.
- $\alpha = \liminf_{j \to \infty} E_K[\gamma_{k_j}] \geq E_K[\hat{\gamma}] \geq \alpha$. QED

Theorem (Lower semicontinuity of E_K) If $c : X \times Y \to [0, \infty]$ is lower semi continuous, then $E_K : \mathcal{A}(\mu, \nu) \to [0, \infty]$ is lower semicontinuous in $\mathcal{P}(X \times Y)$ w.r.t. the narrow topology (i.e., the weak-* topology of $[C_b(X \times Y)]^*$).

Proof Construct $c_k \in \mathrm{Lip}_b(X \times Y)$, $c_k \uparrow c$:

$$c_k(x, y) = \inf_{x' \in X, y' \in Y} \{ c(x', y') \wedge k + k d_X(x, x') + k d_Y(y, y') \}.$$

See Lecture 14. QED

Probability measures on a Polish space

Z: Polish, $\mathcal{P}(Z) = \{$all Borel probability measures$\}$.

Narrow convergence: $\mu_k \to \mu$.

$$\int_Z \varphi \, d\mu_k \to \int_Z \varphi \, d\mu \quad \forall \varphi \in C_b(Z).$$

Same as weak-* convergence w.r.t. $[C_b(Z)]^*$ (in general, not the same as the weak-* convergence w.r.t. $[C_0(Z)]^*$ or $[C_c(Z)]^*$).

Note: $\mathcal{M}_+(Z) \subseteq [C_b(Z)]^*$. The weak-* topology w.r.t. $[C_b(Z)]^*$ is called the narrow topology.

Theorem Let Z be a Polish space. Then, endowed with the narrow topology, $\mathcal{P}(Z)$ is a Polish space.

See Ambrosio-Gigli-Savaré (2008): §5.1.

Definition Let Z be a metric space. A finite Borel measure μ on Z is tight, if for any $\varepsilon > 0$, there exists a compact subset $K \subseteq X$ such that $\mu(X \setminus K) < \varepsilon$.

Ulam lemma If Z is a Polish space, then every $\mu \in \mathcal{M}_+(Z)$ is tight.

Proof Let $D = \{x_i\}_{i=1}^{\infty} \subseteq Z$ be dense in Z. $\forall k \in \mathbb{N}$.

$Z \triangleq \bigcup_{i=1}^{\infty} \overline{B(x_i, 2^{-k})} = \lim_{m \to \infty} F_m$, where $F_m = \bigcup_{i=1}^{m} \overline{B(x_i, 2^{-k})}$.

$F_m \subseteq F_{m+1}$ $(m = 1, 2, \cdots)$. Thus $\mu(Z) = \lim_{m \to \infty} \mu(F_m)$. Since $\mu(Z) < \infty$, $\mu(Z \setminus F_m) = \mu(Z) - \mu(F_m) \to 0$. Thus, $\exists n(k) \in \mathbb{N}$, such that

$$\mu\left(Z \setminus \bigcup_{i=1}^{n(k)} \overline{B(x_i, 2^{-k})}\right) = \mu(Z \setminus F_{n(k)}) < \frac{\varepsilon}{2^{k-1}}.$$

Set $K = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n(k)} \overline{B(x_i, 2^{-k})}$. Then K is closed and totally bounded. Since Z is complete, K is compact. Moreover,

$$\mu(Z \setminus K) = \sum_{k=1}^{\infty} \mu\left(Z \setminus \bigcup_{i=1}^{n(k)} \overline{B(x_i, 2^{-k})}\right) < \varepsilon. \quad \underline{\text{QED}}$$

<u>Definition</u> Let Z be a metric space and $\mathcal{F} \subseteq \mathcal{P}(Z)$. We say $\mathcal{F}$ is equi-tight, or just tight, if for any $\varepsilon > 0$, there exists compact $K \subseteq Z$ such that $\mu(Z \setminus K) < \varepsilon$ for every $\mu \in \mathcal{F}$.

<u>Prokhorov Theorem</u> Let (Z, d) be a Polish space and $\mathcal{F} \subseteq \mathcal{M}_+(Z)$. Suppose $\sup_{\mu \in \mathcal{F}} \mu(Z) < \infty$. Then, $\mathcal{F}$ is relatively compact w.r.t. the narrow topology if and only if $\mathcal{F}$ is tight.

We postpone the proof, as we'd like to prove now the following narrow compactness (i.e., the compactness w.r.t. the narrow topology) of $\mathcal{A}(\mu, \nu)$:

<u>Corollary</u> If X and Y are Polish spaces, $\mu \in \mathcal{P}(X)$, and $\nu \in \mathcal{P}(Y)$, then $\mathcal{A}(\mu, \nu)$ is narrowly compact.

<u>Proof</u> We show first that $\mathcal{A}(\mu, \nu)$ is narrowly closed in $\mathcal{M}_+(Z)$. If $\gamma_k \in \mathcal{A}(\mu, \nu) \subseteq \mathcal{P}(X \times Y)$, $\gamma_\infty \in \mathcal{M}_+(X \times Y)$, and $\gamma_k \to \gamma_\infty$ narrowly, then clearly $\gamma_\infty \in \mathcal{P}(X \times Y)$ (choosing

the test function $\varphi \equiv 1$ on $X \times Y$). Thus, we need only to show that $\mathcal{A}(\mu, \nu)$ is narrowly closed in $\mathcal{P}(X \times Y)$. The narrow closedness of $\mathcal{A}(\mu, \nu)$ in $\mathcal{P}(X \times Y)$ now follows from the following: If $\gamma \in \mathcal{P}(X \times Y)$ then $\gamma \in \mathcal{A}(\mu, \nu)$ if and only if the following hold true:

$$\int_X \varphi \, d\mu = \int_{X \times Y} \varphi \, d\gamma \quad \forall \varphi \in C_b(X);$$

$$\int_Y \psi \, d\nu = \int_{X \times Y} \psi \, d\gamma \quad \forall \psi \in C_b(Y).$$

Since $\sup_{\gamma \in \mathcal{A}(\mu, \nu)} \gamma(X \times Y) = 1$, by Prokhorov's Theorem, we need only to show that $\mathcal{A}(\mu, \nu)$ is equi-tight in $\mathcal{P}(X \times Y)$. Let $\varepsilon > 0$, by Ulam's lemma, $\exists$ compact $K \subseteq X$, compact $H \subseteq Y$, s.t. $\mu(X \setminus K) < \varepsilon/2$ and $\nu(Y \setminus H) < \varepsilon/2$. Hence, if $\gamma \in \mathcal{A}(\mu, \nu)$

$$\gamma((X \times Y) \setminus (K \times H)) \le \gamma((X \setminus K) \times Y) + \gamma(X \times (Y \setminus H))$$
$$= \mu(X \setminus K) + \nu(Y \setminus H) < \varepsilon.$$

QED.

<u>Proof of Prokhorov's Thm</u> Assume W.L.O.G. $\mathcal{F} \subseteq \mathcal{P}(Z)$.

(1) $\mathcal{F}$ is equi-tight $\Longrightarrow$ $\mathcal{F}$ is relatively weak-$*$ compact.

Let $\mu_j \in \mathcal{F}$ ($j = 1, \cdots$). Show there exists a subseq. of μ_j that converges in $\mathcal{P}(Z)$.

$\mathcal{F}$ is equi-tight $\Longrightarrow$ $\exists$ compact sets $K_k \uparrow$ s.t.,

$$\omega_k \triangleq \sup_{\mu \in \mathcal{F}} (Z \setminus K_k) \to 0 \text{ as } k \to \infty.$$

Each K_k is compact, $\{\mu_j\}$ is bounded. Hence, by the diagonal argument, $\exists$ subseq. μ_{j_p}, s.t. $\mu_{j_p} \llcorner K_k \in \mathcal{M}_+(K_k)$, and $\mu_{j_p} \to \nu_k$ in $\mathcal{M}_+(K_k)$, $\forall k$. Viewing $\nu_k \in \mathcal{M}_+(Z)$ with $\operatorname{supp} \nu_k \subseteq K_k$, we have $\nu_k \leq \nu_{k+1}$. Moreover, $1-\omega_k \leq (\mu_{j_p} \llcorner K_k)(Z) \leq 1$, hence, $1-\omega_k \leq \nu_k(Z) \leq 1$, and $\nu_k(Z) \to 1$. Thus, defining ν by

$$\nu(B) := \sup_{k \geq 1} \nu_k(B) = \lim_{k\to\infty} \nu_k(B) \quad \forall B \in \mathcal{B}(Z),$$

we have $\nu \in \mathcal{P}(Z)$: $\nu_k \uparrow \nu \Rightarrow \nu$ is additive, also σ-additive. Hence, ν is σ-additive. clearly, $\nu(Z)=1$.

Finally, $\forall \varphi \in C_b(Z)$, $\forall k \geq 1$,

$$\left|\int_Z \varphi \, d(\mu_{j_p} - \nu)\right| \leq \left|\int_Z \varphi \, d\mu_{j_p} - \int_Z \varphi \, d\mu_{j_p} \llcorner K_k\right|$$

$$+ \left|\int_Z \varphi \, d\mu_{j_p} \llcorner K_k - \int_Z \varphi \, d\nu_k\right| + \underbrace{\left|\int_Z \varphi \, d\nu_k - \int_Z \varphi \, d\nu\right|}_{\to 0}.$$

Hence, $\limsup_{p\to\infty} \left|\int_Z \varphi \, d(\mu_{j_p} - \nu)\right| \leq 2 \sup|\phi| \, \omega_k$.

Let $k \to \infty$. We get $\mu_{j_p} \to \nu$ narrowly.

(2) $\mathcal{F}$ is relatively compact $\Rightarrow$ $\mathcal{F}$ is equi-tight.

$\forall \varepsilon > 0$. Let $D = \{x_i\} \subseteq Z$ be dense in Z. Suffice to show that $\forall j \in \mathbb{N}$ $\exists k_j \in \mathbb{N}$ s.t.

$$\mu\left(Z \setminus \bigcup_{i=1}^{k_j} B(x_i, 1/j)\right) \leq 2^{-j} \varepsilon \quad \forall \mu \in \mathcal{F}. \quad (*)$$

Indeed, setting $K = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{k_j} \overline{B(x_i, 1/j)} \subseteq Z$, K compact, we have $\mu(Z \setminus K) \leq \varepsilon$ for all $\mu \in \mathcal{F}$.

Now, prove $(*)$ by contradiction. If $(*)$ fails, then $\exists j_0 \in \mathbb{N}$ s.t. $\forall k \in \mathbb{N}$ $\exists \mu_k \in \mathcal{F}$ s.t.

$$\mu_k \Big(Z \setminus \bigcup_{i=1}^{k} B(x_i, \tfrac{1}{j_0}) \Big) > 2^{-j_0} \varepsilon.$$

Compactness $\Longrightarrow$ $\exists$ subseq. $\mu_{k_p} \longrightarrow \mu \in \mathcal{P}(Z)$. $\Longrightarrow$ $\forall k \in \mathbb{N}$:

$$\mu\Big(Z \setminus \bigcup_{i=1}^{k} B(x_i, \tfrac{1}{j_0}) \Big) \geq \limsup_{p \to \infty} \mu_{k_p} \Big(Z \setminus \bigcup_{i=1}^{k} B(x_i, \tfrac{1}{j_0}) \Big) \geq 2^{-j_0} \varepsilon,$$

a contradiction, as $\lim_{k \to \infty} \mu(Z \setminus \bigcup_{i=1}^{k} B(x_i, \tfrac{1}{j_0})) = 0$. <u>QED</u>